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Reasoning within Fuzzy OWL 2 EL revisited

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Abstract

Description Logics (DLs) are logics with interesting representational and computational features and are at the core of the *Web Ontology Language* OWL 2 and its profiles among which there is OWL 2 EL. The main feature of OWL 2 EL is that instance/subsumption checking can be decided in polynomial time. On the other hand, *fuzzy DLs* have been proposed as an extension to classical DLs with the aim of dealing with *fuzzy* concepts and we focus here on Fuzzy OWL 2 EL under standard and Gödel semantics. We provide some reasoning algorithms showing that instance/subsumption checking decision problems remain polynomial time for Fuzzy OWL 2 EL. We also identify some issues in previous related work (essentially incompleteness problems).

Keywords: Fuzzy ontologies; Fuzzy description logics; Tractable reasoning

1. Introduction

Description Logics (DLs for short) [5] is a well-known family of logics for knowledge representation, with various representational and computational characteristics. In the last decades, DLs have gained popularity due to their close connection with the *Web Ontology Language* OWL 2 [24], and its profiles OWL 2 QL [79], OWL 2 EL [78] and OWL 2 RL [80].

On the other hand, *Fuzzy DLs* (FDLs) have been proposed as an extension to classical DLs with the aim to deal with *fuzzy* concepts. In these logics, axioms can be satisfied to some degree of truth (typically, a truth value in [0, 1]). There is quite some work on fuzzy DLs in the literature (see e.g., [12,66,68,69] for an overview). Some examples of applications of fuzzy DLs are the Semantic Web [62], recommendation systems [23], image interpretation [25], ambient intelligence [26], diabetes diagnosis [28], and robotics [27].

In this paper, we will focus on the fuzzy counterpart of OWL 2 EL, which is based on the $\mathcal{EL}$ family of DLs [1,3,22,36] and that has as main feature polynomial time reasoning algorithms (e.g., for instance/subsumption checking). We recap that, in the fuzzy setting, there has already been a notable amount of work about the fuzzy $\mathcal{EL}$ family [4,8–11,17,20,21,30,42,43,45,49,56,66,70,72,73,75–77,82] (see Section 4 for a comparison with related work), though

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none of them cover completely Fuzzy OWL 2 EL and are complete inferentially. Here, we further advance the state of the art in this context, especially [49], by addressing all salient parts of Fuzzy OWL 2 EL, *i.e.*, fuzzy $\mathcal{EL}_{\bot}^{+}$ with nominals, fuzzy concrete domains, domain, range and reflexive role restrictions. Specifically, the contributions of this paper are the following ones:

- we address both the case of Gödel logic as well as standard fuzzy logic semantics;
- we allow reasoning with domain, range and reflexive role restrictions;
- we show that we may reduce the so-called nominal safe ontologies to ontologies without nominals. This not only simplifies the inference rules set, but, as it happens for the crisp case, we expect it has a significant practical impact with respect to computation time;
- we address further fuzzy concrete domains by extending usefully the notion of fuzzy p-admissible concrete domains;
- we address the general case (for non-nominal safe ontologies) of nominals;
- we show that for all cases above subsumption checking can be decided in polynomial time;
- we identify some issues in related previous work (essentially incompleteness problems).

The rest of this manuscript is organized as follows. Section 2 starts by providing some background on fuzzy logic and the crisp DLs that will be considered throughout this paper. Then, Section 3 discusses reasoning algorithms for fuzzy logics of the $\mathcal{EL}$ family. Finally, Section 4 overviews some related work and Section 5 sets out some conclusions and ideas for future work. All salient proofs are in the Appendix.

2. Preliminaries

This section quickly overviews the main results on fuzzy logic (Section 2.1) and the classical $\mathcal{EL}$ family (Section 2.2) that will be needed to follow this paper.

2.1. Fuzzy sets and fuzzy logic

Fuzzy set theory and fuzzy logic were proposed by L. A. Zadeh [81] to manage imprecise and vague knowledge. While in classical set theory elements either belong to a set or not, in fuzzy set theory elements can belong to some degree. More formally, let X be a set of elements called the reference set. A *fuzzy subset* A of X is characterized by a membership function $\mu_A(x)$, or simply $A(x)$, which assigns to every $x \in X$ a degree of truth, measured as a value in a truth space $\mathcal{N}$. The truth space is usually $\mathcal{N} = [0, 1]$, but other choices are possible. Indeed, $\mathcal{N}$ does not need to be a total order, nor does it need to be infinite. As in the classical case, 0 means no-membership and 1 full membership, but now a value between 0 and 1 represents the extent to which x can be considered as an element of the fuzzy set A. To distinguish between fuzzy sets and classical (non-fuzzy) sets, we refer to the latter as *crisp sets*.

The definition of the membership function and its shape may depend on the context and may be subjective. However, the trapezoidal (Fig. 1 (a)), the triangular (Fig. 1 (b)), the L-function (left-shoulder function, Fig. 1 (c)), and the R-function (right-shoulder function, Fig. 1 (d)) are simple, but most frequently used to specify membership degrees.

Fuzzy logics provide compositional calculi of degrees of truth. The conjunction, disjunction, complement and implication operations are performed in the fuzzy case by a *t-norm* function $\otimes$, a *t-conorm* function $\oplus$, a *negation* function $\ominus$ and an *implication* function $\Rightarrow$, respectively. For a formal definition of these functions we refer the reader to [32,38].

A quadruple composed by a t-norm, a t-conorm, an implication function and a negation function determines a *fuzzy logic*. One usually distinguishes three fuzzy logics, namely Łukasiewicz, Gödel, and Product [32], due to the fact that any continuous t-norm can be obtained as a combination of Łukasiewicz, Gödel, and Product t-norms [53]. In the following, we consider also what we call *Standard Fuzzy Logic* (SFL) that includes the conjunction, disjunction, and negation originally proposed by Zadeh [81] together with Gaines-Rescher implication.

Table 1 summarizes the fuzzy operators for the four fuzzy logics. Different fuzzy logics satisfy different logical properties (for a detailed list of properties, the reader is referred to [46]). In this paper, we will mainly focus on Gödel fuzzy logic. We will strongly use the fact that Gödel t-norm is *idempotent*, *i.e.*, $\alpha \otimes \alpha = \alpha$ for any $\alpha \in [0, 1]$, and

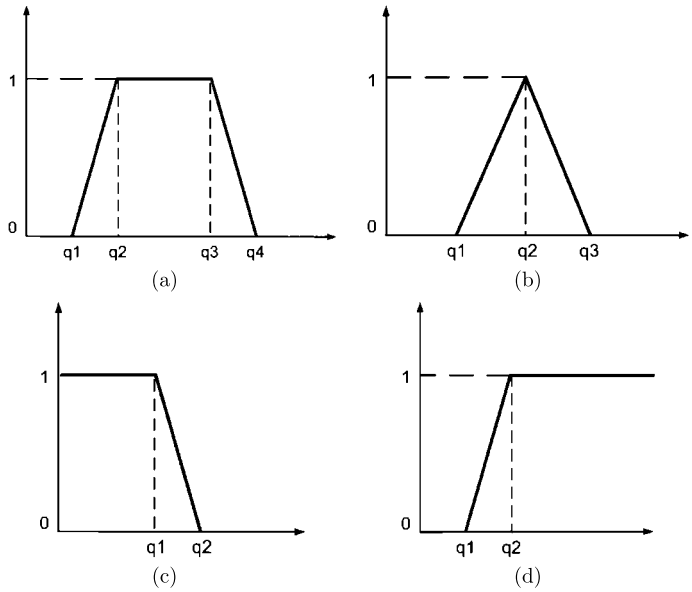

Fig. 1. (a) Trapezoidal function $trz(q_1, q_2, q_3, q_4)$; (b) Triangular function $tri(q_1, q_2, q_3)$; (c) L-function $ls(q_1, q_2)$; and (d) R-function $rs(q_1, q_2)$.

Table 1
Combination functions of various fuzzy logics.

	Gödel logic	Łukasiewicz logic	Product logic	SFL
$\alpha \otimes \beta$	$\min(\alpha, \beta)$	$\max(\alpha + \beta - 1, 0)$	$\alpha \cdot \beta$	$\min(\alpha, \beta)$
$\alpha \oplus \beta$	$\max(\alpha, \beta)$	$\min(\alpha + \beta, 1)$	$\alpha + \beta - \alpha \cdot \beta$	$\max(\alpha, \beta)$
$\alpha \Rightarrow \beta$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta & \text{otherwise} \end{cases}$	$\min(1 - \alpha + \beta, 1)$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ \beta/\alpha & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha \leq \beta \\ 0 & \text{otherwise} \end{cases}$
$\ominus \alpha$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$1 - \alpha$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$1 - \alpha$

that Gödel implication $\Rightarrow$ is an *R-implication*, that is, it is defined as the residuum of its t-norm $\otimes$: $\alpha \Rightarrow \beta = \sup\{\gamma \mid \alpha \otimes \gamma \leq \beta\}$.

Each t-norm and its residuum satisfy the following properties

- $\alpha \Rightarrow \beta = 1$ iff $\alpha \leq \beta$ (ordering property),
- $\alpha \Rightarrow \beta \geq \gamma$ iff $\beta \geq \alpha \otimes \gamma$, and
- from $\alpha \Rightarrow \beta \geq \gamma$ and $\alpha \geq \delta$ we can use fuzzy modus ponens to infer $\beta \geq \gamma \otimes \delta$.

Relations can also be extended by considering fuzzy subsets of the Cartesian product over some reference sets. A (binary) *fuzzy relation* R over two countable sets X and Y is a function $R\colon X \times Y \to [0, 1]$. Several properties of the relations (such as reflexive, irreflexive, symmetric, asymmetric, transitive, or disjoint with another relation) and operations (inverse, composition) can be easily extended to the fuzzy case. In particular, the *composition* of two fuzzy relations $R_1\colon X \times X \to [0, 1]$ and $R_2\colon X \times X \to [0, 1]$ is defined as $(R_1 \circ R_2)(x, z) = \sup_{y \in X} R_1(x, y) \otimes R_2(y, z)$.

2.2. On the $\mathcal{EL}$ family

We recap here the various DL languages of the $\mathcal{EL}$ family we are considering here (for more insights, we refer the reader to [1,3,6,7,22,35,36]).

Table 2
Syntax and semantics of $\mathcal{EL}\mathcal{O}_{\bot}^{+}$.

	Syntax	Semantics
Individuals		
Individual	a	$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
Roles		
Atomic role	r	$r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
Role chain	$r_1 \circ r_2$	$\{\langle x, y\rangle \mid \exists z \in \Delta^{\mathcal{I}}$ s. t. $\langle x, z\rangle \in {r_1}^{\mathcal{I}}$ and $\langle z, y\rangle \in {r_2}^{\mathcal{I}}\}$
Concepts		
Atomic concept	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
Top	$\top$	$\Delta^{\mathcal{I}}$
Bottom	$\bot$	$\emptyset$
Conjunction	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
Existential restriction	$\exists r.C$	$\{x \in \Delta^{\mathcal{I}} \mid \exists y \in \Delta^{\mathcal{I}}$ s. t. $\langle x, y\rangle \in r^{\mathcal{I}}$ and $y \in C^{\mathcal{I}}\}$
Nominal	$\{a\}$	$\{a^{\mathcal{I}}\}$
Axioms		
Concept inclusion	$C \sqsubseteq D$	$C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$
Role inclusion	$r \sqsubseteq s$	$r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$
Role composition	$r_1 \circ r_2 \sqsubseteq s$	$(r_1 \circ r_2)^{\mathcal{I}} \subseteq s^{\mathcal{I}}$
Concept assertion	$C(a)$	$a^{\mathcal{I}} \in C^{\mathcal{I}}$
Role assertion	$r(a, b)$	$\langle a^{\mathcal{I}}, b^{\mathcal{I}}\rangle \in r^{\mathcal{I}}$

Syntax The vocabulary is given by a set of *atomic concepts* (or *concept names*) $\mathsf{N}_{\mathscr{C}} = \{A_1, A_2, \dots\}$, a set of *atomic roles* $\mathsf{N}_{\mathscr{R}} = \{r_1, r_2, \dots, s_1, s_2, \dots\}$ and a set of *individuals* $\mathsf{N}_{\mathscr{O}} = \{a, b, c, \dots\}$. All these sets are assumed to be enumerable and pairwise disjoint.

To start with, $\mathcal{EL}$ *concept expressions* $C, D, \dots$ are built according to the following syntax:

$$C, D \rightarrow A \mid \top \mid C \sqcap D \mid \exists r.C \ .$$

An *ontology* $\mathcal{O}$ is a finite set of *axioms*. A *General Concept Inclusion* (GCI) axiom is of the form $C \sqsubseteq D$ (C *is subsumed by* D), meaning that all instances of concept C are also instances of concept D. We use the expression $C = D$ as a shorthand for having both $C \sqsubseteq D$ and $D \sqsubseteq C$.

On the other hand, *concepts, roles* and *axioms* of the extension of $\mathcal{EL}$ called $\mathcal{EL}\mathcal{O}_{\bot}^{+}$ are defined recursively as in Table 2 [36].

Remark 1 *(Some languages of the $\mathcal{EL}$ family).* The logic $\mathcal{EL}^{+}$ is the extension of $\mathcal{EL}$ with role inclusion and composition axioms [6].

The subscript $\bot$ denotes the fact that the bottom concept is supported. So, e.g. $\mathcal{EL}_{\bot}^{+}$ is the DL $\mathcal{EL}^{+}$ with the addition of the bottom concept [36].

The use of nominal concepts is denoted with the letter $\mathcal{O}$ in the DL literature [5] and, thus, e.g. $\mathcal{EL}\mathcal{O}_{\bot}$ is $\mathcal{EL}_{\bot}$ extended with nominal concepts, while $\mathcal{EL}\mathcal{O}_{\bot}^{+}$ is $\mathcal{EL}_{\bot}^{+}$ with nominal concepts [36].

When there are individuals, it is possible to have concept and role assertions. A *concept assertion* axiom is of the form $C(a)$ dictating that individual a is instance of concept C. A *role assertion* is an expression of the form $r(a, b)$, dictating that individual a is related to individual b via role r.

It is possible to extend the logic with concrete domains, indicated with the letter **D**. The language $\mathcal{EL}\mathcal{O}_{\bot}^{+}(\mathbf{D})$ is also called $\mathcal{EL}^{++}$ [1]. Even after extending the logic with range and reflexive role restrictions (see Section 2.2.2 later on), the same name is still used [3] and the latter is indeed the core of the OWL EL specification [78].

Example 2. $\mathcal{O}_1 = \{A \sqsubseteq \exists s.C \sqcap \top \sqcap \exists s.\{a\}, C \sqsubseteq B, \exists s.B \sqsubseteq B\}$ is an $\mathcal{EL}\mathcal{O}_{\bot}^{+}$ ontology with three axioms. $\mathcal{O}_2 = \mathcal{O}_1 \cup \{C \sqsubseteq \{a\}\}$ extends it with a fourth axiom.

Remark 3. Three important types of axioms are representable in $\mathcal{EL}_{\bot}^{+}$ or more expressive logics:

- a *concept disjointness* axiom can be expressed as $C_1 \sqcap C_2 \sqsubseteq \bot$;

- a *complex role composition* $r_1 \circ r_2 \circ \cdots \circ r_n \sqsubseteq s$ is a shorthand for the set of axioms $r_1 \circ r_2 \sqsubseteq r_{x_1}$, $r_{x_1} \circ r_3 \sqsubseteq r_{x_2}$, $\ldots$; $r_{x_{n-2}} \circ r_n \sqsubseteq s$;
- finally, a *transitive role* axiom can be represented as $r \circ r \sqsubseteq r$.

Note also that a role inclusion $r \sqsubseteq s$ is a special case of role composition with the identity role as one of the roles in the left side (see Section 2.2.2 later on).

A *nominal safe* language $\mathcal{L}$ of the $\mathcal{EL}$ family is the language $\mathcal{L}$ with some restrictions on the occurrence of nominals and is defined as follows [36]. A $\mathcal{L}$ concept C is *safe* if C has only occurrences of nominals in sub concepts of the form $\exists r.\{a\}$; C is *negatively safe* (in short, *n-safe*) if C is either safe or a nominal. A GCI $C \sqsubseteq D$ is *safe* if C is n-safe and D is safe. An $\mathcal{L}$ ontology is *nominal safe* if all its GCIs are safe. It is worth noting that nominal safeness is a quite commonly used pattern of nominals in OWL 2 EL ontologies [36].

Example 4. Revisiting again the ontologies in Example 2, the ontology $\mathcal{O}_1$ is nominal safe, while $\mathcal{O}_2$ is not.

In the following, let $\mathcal{O}$ be an $\mathcal{ELO}_\bot^+$ ontology. Let $\mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$ be the set of atomic concepts and nominal concepts occurring in $\mathcal{O}$. We say that $\mathcal{O}$ is *normal form* if all axioms in it are, *i.e.*, each axiom has one of the following forms ($n \geq 2$)

$$A \sqsubseteq B \ , \ A_1 \sqcap \ldots \sqcap A_n \sqsubseteq B \ , \ A \sqsubseteq \exists r.B \ , \ \exists r.A \sqsubseteq B \ , \ r \sqsubseteq s \ , \ r_1 \circ r_2 \sqsubseteq s \ ,$$

where $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\top, \bot\}$ and $A_i \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$.

Semantics An *interpretation* is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set, called *interpretation domain* and $\cdot^{\mathcal{I}}$ is an *interpretation function*

1. mapping atomic concepts A into subsets $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$;
2. mapping $\top$ into $\top^{\mathcal{I}} = \Delta^{\mathcal{I}}$;
3. mapping roles r into a subset $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$;
4. mapping each individual $a \in \mathsf{N}_{\mathscr{O}}$ into an object $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

The interpretation function $\cdot^{\mathcal{I}}$ is extended to complex concept expressions as shown in Table 2.

An interpretation $\mathcal{I}$ *satisfies* (is a *model* of) an axiom τ, denoted $\mathcal{I} \models \tau$, if the corresponding condition in Table 2 holds. $\mathcal{I}$ *satisfies* (is a *model* of) an ontology $\mathcal{O}$, denoted $\mathcal{I} \models \mathcal{O}$, if $\mathcal{I}$ satisfies each axiom in $\mathcal{O}$. An ontology $\mathcal{O}$ is *consistent* if it has a model, otherwise it is *inconsistent*. An axiom τ is *entailed* by an $\mathcal{O}$, denoted as $\mathcal{O} \models \tau$, if every model of $\mathcal{O}$ is a model of τ. A concept C is *unsatisfiable* w.r.t. $\mathcal{O}$ if $\mathcal{O} \models C \sqsubseteq \bot$, otherwise C is *satisfiable* w.r.t. $\mathcal{O}$. A concept C is *subsumed* by concept D w.r.t. $\mathcal{O}$ if $\mathcal{O} \models C \sqsubseteq D$. Concepts C, D are *equivalent* w.r.t. $\mathcal{O}$ if $\mathcal{O} \models C = D$. An individual a is *instance* of a concept C w.r.t. $\mathcal{O}$ if $\mathcal{O} \models C(a)$.

Remark 5. Note that concept assertions $C(a)$ and role assertions $r(a, b)$ can easily be represented in e.g. nominal safe $\mathcal{ELO}_\bot^+$ via the mapping $C(a) \mapsto \{a\} \sqsubseteq C$ and $r(a, b) \mapsto \{a\} \sqsubseteq \exists r.\{b\}$.

In the following, we recap here some salient facts related to nominal safe $\mathcal{ELO}_\bot^+$ [36, Appendix A]. That is, one can replace nominals in a nominal safe $\mathcal{ELO}_\bot^+$ ontology $\mathcal{O}$ with newly introduced concept names, yielding an $\mathcal{EL}_\bot^+$ ontology $\mathcal{O}'$, such that $\mathcal{O}'$ supports the same entailments as $\mathcal{O}$. Hence, an entailment decision procedure for $\mathcal{EL}_\bot^+$ suffices to decide entailment for nominal safe $\mathcal{ELO}_\bot^+$ (but not for unrestricted $\mathcal{ELO}_\bot^+$). Please note that although there is a more general procedure that works even for non-nominal safe ontologies [35] (see also Section 3.5), the case of nominal safe ontologies is very interesting from a practical point of view: (i) many ontologies are nominal safe; and (ii) the specialized rules for nominal safe ontologies are much more efficient than those for the general case.[1]

Specifically, consider an $\mathcal{ELO}_\bot^+$ ontology $\mathcal{O}$. For each individual a occurring in $\mathcal{O}$ consider a new atomic concept N_a. For x an $\mathcal{ELO}_\bot^+$ concept, GCI, or ontology, we define $N(x)$ to be the result of replacing each occurrence of each nominal $\{a\}$ in x with N_a. The following proposition provides a sufficient condition to check entailment.

[1] Indeed, it is one of the optimizations implemented in the efficient crisp DL reasoner ELK [36].

Proposition 6 *([36], Lemma 5 and Corollary 2). Let $\mathcal{O}$ be an $\mathcal{ELO}_{\bot}^{+}$ ontology and τ an $\mathcal{ELO}_{\bot}^{+}$ axiom that do not contain atomic concepts of the form N_a. Then*

1. *if $N(\mathcal{O}) \models N(\tau)$ then $\mathcal{O} \models \tau$;*
2. *if $N(\mathcal{O}) \models N_a \sqsubseteq \bot$ for some a then $\mathcal{O}$ is inconsistent.*

The converse of Proposition 6 does not hold in general, but holds for nominal safe $\mathcal{ELO}_{\bot}^{+}$.

Proposition 7 *([36], Theorem 4). Let $\mathcal{O}$ be a nominal safe $\mathcal{ELO}_{\bot}^{+}$ ontology and τ a safe $\mathcal{ELO}_{\bot}^{+}$ GCI that do not contain atomic concepts of the form N_a. Assume $N(\mathcal{O}) \not\models N_a \sqsubseteq \bot$ for all a. Then*

1. *$\mathcal{O}$ is consistent;*
2. *if $\mathcal{O} \models \tau$ then $N(\mathcal{O}) \models N(\tau)$.*

Note that Proposition 7 fails if the use of nominals is not safe (see [36], Remark 2] for an example).

2.2.1. Reasoning in $\mathcal{EL}_{\bot}^{+}$ and nominal safe $\mathcal{ELO}_{\bot}^{+}$

Here we provide the inference rules for the case of assertion free $\mathcal{EL}_{\bot}^{+}$. Since by Propositions 6–7 and Remark 5 any nominal safe $\mathcal{ELO}_{\bot}^{+}$ ontology can be reduced to an assertion free $\mathcal{EL}_{\bot}^{+}$ ontology, this also covers nominal safe $\mathcal{ELO}_{\bot}^{+}$.

Remark 8. Please, note that in [39–41] and [35, Remark 6] it is shown that the calculus for $\mathcal{EL}^{++}$, and in particular for $\mathcal{ELO}$, as illustrated in [1,3], is incomplete in the presence of unrestricted use of nominals. For instance, given the ontology [35] (but see also [40] for other examples),

$$\mathcal{O} = \{A \sqsubseteq \exists r.(B \sqcap \{a\}), A \sqsubseteq \exists s.\{a\}, \exists s.B \sqsubseteq B\} ,$$

then $\mathcal{O} \models A \sqsubseteq B$, because if A is not empty then a is an instance of B. However, the algorithm in [1,3] is not able to infer it. For further insights we refer the reader to [34,35,39–41,48], and specifically to [34,35] for a complete and similar to [1,3] calculus for the case of unrestricted use of nominals. For a Datalog based calculus, we refer the reader to [39–41].

A general reasoning problem is checking entailment, *i.e.*, given an ontology $\mathcal{O}$ and an axiom τ, check if $\mathcal{O} \models \tau$ holds. However, a major problem consists of the *ontology classification* problem that is the task to compute the *taxonomy* representing all entailed subsumption and equivalences between $\bot$, $\top$, and atomic concepts occurring in $\mathcal{O}$. This is justified e.g. by the fact that $\mathcal{O} \models C \sqsubseteq D$ iff $\mathcal{O} \cup \{A \sqsubseteq C, D \sqsubseteq B\} \models A \sqsubseteq B$, where A, B are new atomic concepts.

So, let $\mathcal{O}$ be an assertion free $\mathcal{EL}_{\bot}^{+}$ ontology. In the following, we present a simple procedure to determine all subsumption relationships among atomic concepts in $\mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$, inspired by [1–3,36].

To start with, Fig. 2 shows the inference rules to transform an $\mathcal{ELO}_{\bot}^{+}$ ontology, and thus also $\mathcal{O}$, into normal form (in particular, see [2, Figure 1]). The precondition of the rule is above the horizontal line, while its conclusion is below the line. The label in front of the rule is its name and the side conditions are after the rule.

Example 9. Given the ontology $\mathcal{O}_1$ from Example 2,

$$N(\mathcal{O}_1) = \{A \sqsubseteq \exists s.C \sqcap \top \sqcap \exists s.N_a, C \sqsubseteq B, \exists s.B \sqsubseteq B\} .$$

The transformation of $N(\mathcal{O}_1)$ into normal form produces, via rule N_5,

$$\mathcal{O}_1' = \{A \sqsubseteq \exists s.C, A \sqsubseteq \top, A \sqsubseteq \exists s.N_a, C \sqsubseteq B, \exists s.B \sqsubseteq B\} .$$

The inference rules over an $\mathcal{EL}_{\bot}^{+}$ ontology in normal form are depicted in Fig. 3 instead. Let $\mathsf{closure}(\mathcal{O})$ be a set that is closed under the rules in Figs. 2 and 3 applied to $\mathcal{O}$.

$(N_0^{\bot})$ $\frac{C_1 \sqcap \bot \sqcap C_2 \sqsubseteq D}{\bot \sqsubseteq D}$:

$(N_0^{\top})$ $\frac{C_1 \sqcap \top \sqcap C_2 \sqsubseteq D}{C_1 \sqcap C_2 \sqsubseteq D}$:

(N_1) $\frac{C_1 \sqcap C \sqcap C_2 \sqsubseteq D}{C \sqsubseteq A \ , \ C_1 \sqcap A \sqcap C_2 \sqsubseteq D}$: $C \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, A new concept name

(N_2) $\frac{\exists r.C \sqsubseteq D}{C \sqsubseteq A \ \exists r.A \sqsubseteq D}$: $C \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$, A new concept name

(N_3) $\frac{C \sqsubseteq D}{C \sqsubseteq A \ A \sqsubseteq D}$: $C, D \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, A new concept name

(N_4) $\frac{B \sqsubseteq \exists r.C}{B \sqsubseteq \exists r.A \ A \sqsubseteq C}$: $C \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$, A new concept name

(N_5) $\frac{C \sqsubseteq D_1 \sqcap D_2}{C \sqsubseteq D_1 \ C \sqsubseteq D_2}$:

Fig. 2. Normal form transformation rules for $\mathcal{ELO}_{\bot}^{+}$.

(R_0) $\frac{}{A \sqsubseteq A}$: $A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

(R_1) $\frac{}{C \sqsubseteq \top}$: $C \sqsubseteq \top$ in normal form

(R_2) $\frac{}{\tau}$: $\tau \in \mathcal{O}$

$(R_{\sqsubseteq}^1)$ $\frac{C \sqsubseteq D' \ D' \sqsubseteq D}{C \sqsubseteq D}$:

$(R_{\sqsubseteq}^2)$ $\frac{A \sqsubseteq B_1 ... A \sqsubseteq B_n \ B_1 \sqcap ... \sqcap B_n \sqsubseteq B}{A \sqsubseteq B}$: $A, B, B_i \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

$(R_{\exists})$ $\frac{A \sqsubseteq \exists r.B \ B \sqsubseteq C}{A \sqsubseteq \exists r.C}$: $A, B, C \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $C \neq \bot$

$(R_{\bot})$ $\frac{A \sqsubseteq \exists r.B \ B \sqsubseteq \bot}{A \sqsubseteq \bot}$: $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

(R_H) $\frac{A \sqsubseteq \exists r.B \ r \sqsubseteq s}{A \sqsubseteq \exists s.B}$:

$(R_{H^\star})$ $\frac{r_1 \sqsubseteq r_2 \ r_2 \sqsubseteq r_3}{r_1 \sqsubseteq r_3}$:

$(R_{\circ})$ $\frac{A_1 \sqsubseteq \exists r_1.B \ B \sqsubseteq \exists r_2.A_2 \ r_1 \circ r_2 \sqsubseteq s}{A_1 \sqsubseteq \exists s.A_2}$: $A_1, B, A_2 \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

Fig. 3. Inference rules for $\mathcal{EL}_{\bot}^{+}$.

Example 10. Given the ontology $N(\mathcal{O}_1)$ from Example 9, $\mathsf{closure}(N(\mathcal{O}_1))$ contains the following axioms:

(1) $A \sqsubseteq \exists s.C$

(2) $A \sqsubseteq \top$

(3) $A \sqsubseteq \exists s.N_a$

(4) $C \sqsubseteq B$

(5) $\exists s.B \sqsubseteq B$

(6) $A \sqsubseteq A$

(7) $B \sqsubseteq B$

(8) $C \sqsubseteq C$

(9) $N_a \sqsubseteq N_a$

(10) $B \sqsubseteq \top$

(11) $C \sqsubseteq \top$

Table 3
Syntax and semantics of range, domain and reflexive role restrictions.

	Syntax	Semantics
Axioms		
Domain restriction	$\mathsf{dom}(r) \sqsubseteq A$	$r^{\mathcal{I}} \subseteq A^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
Range restriction	$\mathsf{ran}(r) \sqsubseteq A$	$r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times A^{\mathcal{I}}$
Reflexive role restriction	$\epsilon \sqsubseteq r$	$\epsilon^{\mathcal{I}} \subseteq r^{\mathcal{I}}$

(12) $N_a \sqsubseteq \top$

(13) $\exists s.B \sqsubseteq \top$

(14) $\exists s.C \sqsubseteq \top$

(15) $\exists s.N_a \sqsubseteq \top$

(16) $A \sqsubseteq \exists s.B$

(17) $A \sqsubseteq B$.

Axioms (1)–(5) come from the original ontology (rule R_2), rule R_0 added (6)–(9), R_1 added (10)–(15), rule $R_\exists$ added (16), and, finally, $R^1_\sqsubseteq$ added (17).

The following follows from [1–3,36].[2]

Proposition 11. *Let $\mathcal{O}$ be an assertion free $\mathcal{EL}^+_\bot$ ontology and let $A, B \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}$. Then*

1. $\mathsf{closure}(\mathcal{O})$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.
2. $\mathcal{O} \models A \sqsubseteq B$ *iff* $A \sqsubseteq B \in \mathsf{closure}(\mathcal{O})$ *or* $A \sqsubseteq \bot \in \mathsf{closure}(\mathcal{O})$.

Remark 12. Please note that in Proposition 11, the concept names A and B have to occur in the ontology $\mathcal{O}$ and, thus, cannot be chosen among the newly introduced ones, as e.g. by rule (N_3).

By Propositions 6 and 7, and by Remark 5, we have

Corollary 13. *Proposition 11 holds also for nominal safe $\mathcal{ELO}^+_\bot$ ontologies.*

We conclude with the following remark.

Remark 14. If we restrict our attention to concept subsumption then rule $R_{H^\star}$ is in fact not needed.

2.2.2. Reasoning with domain, range and reflexive role restrictions

Following [3], we next recap that one may extend $\mathcal{EL}^+_\bot$ further with reflexive role, range and domain restrictions, and show that subsumption remains tractable if a certain syntactic restriction is adopted on role composition axioms. We will adapt this approach in Section 3.3 to the fuzzy case.

To this end, the syntax and semantic conditions of *domain restrictions*, *range restrictions* and *reflexive role restrictions* are described in Table 3, where (i) ϵ is the *identity role* whose extension is $\epsilon^{\mathcal{I}} = \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$, for all interpretations $\mathcal{I}$; and (ii) A is an atomic concept. According to [3], to avoid intractability (and even undecidability), we have to impose a restriction on the structure of an ontology that prevents the otherwise too intricate interplay of range restrictions and role inclusions. Specifically, for an ontology $\mathcal{O}$, roles r, s, we write $\mathcal{O} \vdash r \sqsubseteq s$ iff $r = s$ or $\mathcal{O}$ contains role inclusions

$$r_1 \sqsubseteq r_2, \ldots, r_{n-1} \sqsubseteq r_n \text{ with } r = r_1 \text{ and } s = r_n \ .$$

[2] In particular, see Lemmas 5 (complexity), 6 (soundness), and 7 (completeness) in [2].

Furthermore, we write $\mathcal{O} \vdash \mathsf{ran}(r) \sqsubseteq A$ if there is a role s with $\mathcal{O} \vdash r \sqsubseteq s$ and $\mathsf{ran}(s) \sqsubseteq A \in \mathcal{O}$. Now, the mentioned restriction is as follows:

($\star$) If $r_1 \circ \ldots \circ r_n \sqsubseteq s \in \mathcal{O}$ with $1 \leq n \leq 2$ and $\mathcal{O} \vdash \mathsf{ran}(s) \sqsubseteq A$, then $\mathcal{O} \vdash \mathsf{ran}(r_n) \sqsubseteq A$.

The restriction ensures that if a role inclusion $r_1 \circ \ldots \circ r_n \sqsubseteq s \in \mathcal{O}$ implies a role relationship $\langle x, y \rangle \in s^{\mathcal{I}}$, then the range restriction on s does not impose new concept memberships of y.

Remark 15. Note that the condition ($\star$) is obviously true if the role inclusion is a reflexive role restriction, a role hierarchy statement, or a transitivity statement.

Remark 16. OWL 2 EL satisfies the above-mentioned condition ($\star$) [78].

Reasoning We next show how to deal with these additional axioms in order to decide concept subsumption.

Remark 17. At first, let us note that an axiom $\mathsf{dom}(r) \sqsubseteq A$ can be replaced, *w.l.o.g.*, with a concept inclusion axiom $\exists r.\top \sqsubseteq A$ (in normal form) and, thus, we have not to deal with domain restrictions further. Note also that for more expressive DLs, such as $\mathcal{ALC}$ [5], range restrictions can be replaced with the $\mathcal{ALC}$ concept inclusion $\top \sqsubseteq \forall r.A$, which however is not an $\mathcal{EL}_{\bot}^{+}$ axiom.

Reflexive role axioms are dealt with by adapting rules R_H, $R_{H^\star}$, and $R_\circ$ to consider the case $\epsilon \sqsubseteq r$ as well.

We next show how to remove range restriction axioms. Let $\mathcal{O}$ be an assertion free $\mathcal{EL}_{\bot}^{+}$ ontology, which may contain range and reflexive role restriction axioms. Let us assume that $\mathcal{O}$ is in normal form.[3] We proceed as follows.

1. For each role r, let $\mathsf{ran}_{\mathcal{O}}(r) = \{A \mid \mathcal{O} \vdash \mathsf{ran}(r) \sqsubseteq A\}$.
2. For each $C \sqsubseteq \exists r.B \in \mathcal{O}$, introduce a new atomic concept $X_{r,B}$.
3. Let $\mathcal{O}'$ be obtained from $\mathcal{O}$ by removing all range restriction axioms and performing additionally the following actions:
   (a) exchange every $C \sqsubseteq \exists r.B$ with the axioms $C \sqsubseteq \exists r.X_{r,B}$, $X_{r,B} \sqsubseteq B$, and $X_{r,B} \sqsubseteq A$ for all $A \in \mathsf{ran}_{\mathcal{O}}(r)$;
   (b) if $\epsilon \sqsubseteq r \in \mathcal{O}$, then add $\top \sqsubseteq A$ for all $A \in \mathsf{ran}_{\mathcal{O}}(r)$.

Note that the size of $\mathcal{O}'$ is quadratically bounded by $|\mathcal{O}|$. The following holds.

Proposition 18 *([3]). Let $\mathcal{O}$ be an assertion free $\mathcal{EL}_{\bot}^{+}$ ontology, which may contain range and reflexive role restriction axioms, and $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. Then $\mathcal{O} \models A \sqsubseteq B$ iff $\mathcal{O}' \models A \sqsubseteq B$, where $\mathcal{O}'$ has been determined by the steps above.*

Of course, by Propositions 6 and 7, and by Remark 5 we have that

Corollary 19. *Proposition 18 holds also for nominal safe $\mathcal{ELO}_{\bot}^{+}$ ontologies, which may contain range and reflexive role restriction axioms.*

Therefore, by Proposition 13

Corollary 20. *Subsumption can be determined in polynomial time for nominal safe $\mathcal{ELO}_{\bot}^{+}$ ontologies, even in the presence of domain, range and reflexive role restriction axioms.*

3. On the Fuzzy $\mathcal{EL}$ family

In the following, we introduce the basics of the Fuzzy $\mathcal{EL}$ family, paying special attention to the fuzzy DL $\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$. Conceptually, fuzzy DLs are a fragment of Mathematical Fuzzy Logic with unary and binary predicates

[3] We assume here that axioms of the form $\mathsf{ran}(r) \sqsubseteq A$ and $\epsilon \sqsubseteq r$ are normal forms.

only. However, additionally we also consider so-called *fuzzy concrete domains* [60]), which is one of the main ingredients to make fuzzy DLs useful in practice and part of the Fuzzy OWL 2 language [13].

3.1. Syntax and semantics

Fuzzy concrete domains For the sake of completeness, we recap here the notion of *fuzzy concrete domain*, that is a tuple $\mathbf{D} = \langle \Delta^{\mathbf{D}}, \cdot^{\mathbf{D}} \rangle$ with data type domain $\Delta^{\mathbf{D}}$ and a mapping $\cdot^{\mathbf{D}}$ that assigns to each data value an element of $\Delta^{\mathbf{D}}$, and to every 1-ary *data type predicate* $\mathbf{d}$ a 1-ary fuzzy relation over $\Delta^{\mathbf{D}}$. Therefore, $\cdot^{\mathbf{D}}$ maps indeed each data type predicate into a function from $\Delta^{\mathbf{D}}$ to $[0, 1]$. Note that we restrict to unary datatypes only as they are the only ones supported by crisp OWL 2 EL. In this work, for the sake of ease the presentation, we will also always assume that $\Delta^{\mathbf{D}}$ is the set of rational numbers.[4]

The data type predicates $\mathbf{d}$ we are considering here are well known membership functions supported in Fuzzy OWL 2 [13]:

$$\mathbf{d} \rightarrow ls(q_1, q_2) \mid rs(q_1, q_2) \mid tri(q_1, q_2, q_3) \mid trz(q_1, q_2, q_3, q_4)$$
$$\mid \ \geq_v \ \mid \ \leq_v \ \mid \ =_v \ ,$$

where e.g. ls, rs, tri, and trz represent the left-shoulder, right-shoulder, triangular and trapezoidal membership functions respectively (see Fig. 1) and $v \in \Delta^{\mathbf{D}}$. If fuzzy concrete domains are considered then the letter $\mathbf{D}$ is used in the specification of the language, e.g., $\mathcal{EL}(\mathbf{D})$ is $\mathcal{EL}$ extended with (fuzzy) concrete domains.

Syntax Similarly as in crisp DLs, the elements of fuzzy DLs are concept expressions, roles (which can be *object properties* or *datatype properties*), and individuals. Informally, the range of an object property is a concept, while the range of a datatype property is a datatype predicate. For a datatype property t, we allow concept expressions of the form $\exists t.\mathbf{d}$, where $\mathbf{d}$ is a possibly fuzzy datatype predicate and role t is assumed to be functional and crisp.

A *fuzzy ontology* $\mathcal{O}$ is a finite set of fuzzy axioms. If τ is a crisp axiom and $\alpha \in (0, 1]$ is a rational number, then $\langle \tau, \alpha \rangle$ is a *fuzzy axiom*, dictating that the degree of truth of τ is greater or equal than α. The value α can be omitted and in that case $\alpha = 1$ is assumed.[5] Furthermore, in case τ is a domain, range, reflexive role restriction axiom, or a concept equivalence axiom, we postulate that $\alpha = 1$ (see also [13,15]).

The syntax of fuzzy $\mathcal{ELO}^+_\bot(\mathbf{D})$ is shown in Table 4. Other languages of the fuzzy family are named as in the crisp case (see Remark 1), but in the fuzzy case we allow fuzzy axioms.

Example 21. $\mathcal{O}_3 = \{\langle A \sqsubseteq \exists s.C \sqcap \top \sqcap \exists s.\{a\}, 0.8\rangle, \langle C \sqsubseteq B, 0.7\rangle, \langle \exists s.B \sqsubseteq B, 0.6\rangle\}$ can be seen as a fuzzy version of the ontology $\mathcal{O}_1$ in Example 2.

The notion of *nominal safeness* is as for $\mathcal{ELO}^+_\bot$, where a fuzzy concept inclusion axiom is *safe* if the involved crisp concept inclusion is safe.

For a crisp $\mathcal{ELO}^+_\bot$ axiom τ, we say that $\langle \tau, \alpha \rangle$ is in *normal form* if τ is. We extend the notion of normal form to fuzzy $\mathcal{ELO}^+_\bot(\mathbf{D})$ by extending normal form axioms to the cases:

$$\langle A \sqsubseteq \exists t.\mathbf{d}, \alpha \rangle \ , \ \langle \exists t.\mathbf{d} \sqsubseteq B, \alpha \rangle \ ,$$

where $A, B \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}} \cup \{\bot, \top\}$.

Remark 22. Similarly as in the crisp case (Remark 3), several important types of axioms are representable in $\mathcal{EL}^+_\bot$ or more expressive logics: *concept disjoint* axioms of the form $C_1 \sqcap C_2 \sqsubseteq \bot$, *role composition* of the form $r_1 \circ r_2 \circ \cdots \circ r_n \sqsubseteq s$, and *transitive role* axioms of the form $r \circ r \sqsubseteq r$.

[4] Note that other non-numerical datatypes such as strings or dates can be treated as numbers as well.

[5] In that case, we also omit $\langle$ and $\rangle$.

Table 4
Syntax and semantics of fuzzy $\mathcal{ELO}^{+}_{\bot}(\mathbf{D})$.

	Syntax	Semantics
Individuals		
Individual	a	$a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$
Roles		
Object property	r	$r^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$
Data type property	t	$t^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathbf{D}} \to \{0,1\}$
Role chain	$r \circ s$	$(r \circ s)^{\mathcal{I}}(x,y) = \sup_{z \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,z) \otimes s^{\mathcal{I}}(z,y)\}$
Concepts		
Atomic concept	A	$A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0,1]$
Top	$\top$	$\top^{\mathcal{I}}(x) = 1$
Bottom	$\bot$	$\bot^{\mathcal{I}}(x) = 0$
Conjunction	$C \sqcap D$	$(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$
Object property restriction	$\exists r.C$	$(\exists r.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$
Data property restriction	$\exists t.\mathbf{d}$	$(\exists t.\mathbf{d})^{\mathcal{I}}(x) = \sup_{v \in \Delta^{\mathbf{D}}}\{t^{\mathcal{I}}(x,v) \otimes \mathbf{d}^{\mathbf{D}}(v)\}$
Nominal	$\{a\}$	$\{a\}^{\mathcal{I}}(x) = 1$ if $a^{\mathcal{I}} = x$, else 0
Crisp DL Axioms		
Concept inclusion	$C \sqsubseteq D$	$(C \sqsubseteq D)^{\mathcal{I}} = \inf_{x \in \Delta^{\mathcal{I}}}\{C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)\}$
Role inclusion	$r \sqsubseteq s$	$(r \sqsubseteq s)^{\mathcal{I}} = \inf_{x,y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,y) \Rightarrow s^{\mathcal{I}}(x,y)\}$
Role composition	$r_1 \circ r_2 \sqsubseteq s$	$(r_1 \circ r_2 \sqsubseteq s)^{\mathcal{I}} = \inf_{x,y \in \Delta^{\mathcal{I}}}\{(r_1 \circ r_2)^{\mathcal{I}}(x,y) \Rightarrow s^{\mathcal{I}}(x,y)\}$
Domain restriction	$\mathsf{dom}(r) \sqsubseteq A$	$(\mathsf{dom}(r) \sqsubseteq A)^{\mathcal{I}} = \inf_{x \in \Delta^{\mathcal{I}}}\{\{\sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y)\} \Rightarrow A^{\mathcal{I}}(x)\}$
Range restriction	$\mathsf{ran}(r) \sqsubseteq A$	$(\mathsf{ran}(r) \sqsubseteq A)^{\mathcal{I}} = \inf_{x \in \Delta^{\mathcal{I}}}\{1 \Rightarrow \inf_{y \in \Delta^{\mathcal{I}}}\{r^{\mathcal{I}}(x,y) \Rightarrow A^{\mathcal{I}}(y)\}\}$
Concept assertion	$C(a)$	$(C(a))^{\mathcal{I}} = C^{\mathcal{I}}(a^{\mathcal{I}})$
Role assertion	$r(a,b)$	$(r(a,b))^{\mathcal{I}} = r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$

Semantics Let us fix a fuzzy logic and a fuzzy concrete domain. Unlike classical DLs in which an interpretation $\mathcal{I}$ maps e.g. a concept C into a set of individuals $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, *i.e.*, $\mathcal{I}$ maps C into a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to \{0,1\}$ (either an individual belongs to the extension of C or does not belong to it), in fuzzy DLs, $\mathcal{I}$ maps C into a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$ and, thus, an individual belongs to the extension of C to some degree in $[0,1]$, *i.e.*, $C^{\mathcal{I}}$ is a fuzzy set.

Specifically, a *fuzzy interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a nonempty (crisp) set $\Delta^{\mathcal{I}}$ (the *domain*) and of a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns:

1. to each atomic concept A a function $A^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \to [0,1]$;
2. to $\top$ (resp. $\bot$) the function $\top^{\mathcal{I}}(x) = 1$ (resp. $\bot^{\mathcal{I}}(x) = 0$);
3. to each object property r a function $r^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$;
4. to each data type property t a function $t^{\mathcal{I}}\colon \Delta^{\mathcal{I}} \times \Delta^{\mathbf{D}} \to \{0,1\}$ such that for all $x \in \Delta^{\mathcal{I}}$, for all $v_1, v_2 \in \Delta^{\mathbf{D}}$ if $t^{\mathcal{I}}(x, v_i) = 1$ then $v_1 = v_2$;
5. to each individual a an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$;
6. to each data value v an element $v^{\mathcal{I}} \in \Delta^{\mathbf{D}}$ such that for distinct values v_1 and v_2 we have $v_1^{\mathcal{I}} \neq v_2^{\mathcal{I}}$.

Note that datatype properties are considered as functional and we further assume they are crisp.[6]

The interpretation function $\cdot^{\mathcal{I}}$ is extended to complex concept expressions, role composition and crisp DL axioms as shown in Table 4, and constrains the identity role ϵ such that $\epsilon^{\mathcal{I}}(x,x) = 1$. We further assume that fuzzy interpretations are *witnessed* [33], *i.e.*, the sup and inf are attained at some point of the involved domain. For instance, for all $x \in \Delta^{\mathcal{I}}$ there is $y \in \Delta^{\mathcal{I}}$ such that $(\exists r.C)^{\mathcal{I}}(x) = r^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)$.

A fuzzy interpretation $\mathcal{I}$ *satisfies* (is a *model* of) a fuzzy axiom $\langle \tau, \alpha \rangle$, denoted $\mathcal{I} \models \langle \tau, \alpha \rangle$, if $\tau^{\mathcal{I}} \geq \alpha$, where $\tau^{\mathcal{I}}$ is defined in Table 4. $\mathcal{I}$ *satisfies* (is a *model* of) an ontology $\mathcal{O}$, denoted $\mathcal{I} \models \mathcal{O}$, if $\mathcal{I}$ satisfies each axiom in $\mathcal{O}$. An ontology $\mathcal{O}$ is *consistent* if it has a model, otherwise it is *inconsistent*. An axiom $\langle \tau, \alpha \rangle$ is *entailed* by a $\mathcal{O}$, denoted

[6] So far, no meaningful example emerged requiring the contrary. Note that the reasoner *fuzzyDL* [15] makes this assumption as well.

as $\mathcal{O} \models \langle \tau, \alpha \rangle$, if every model of $\mathcal{O}$ is a model of $\langle \tau, \alpha \rangle$. In that case we say that τ is entailed to degree n w.r.t. $\mathcal{O}$. A concept C is *unsatisfiable* w.r.t. $\mathcal{O}$ if $\mathcal{O} \models C \sqsubseteq \bot$, otherwise C is *satisfiable* w.r.t. $\mathcal{O}$. The *best entailment degree* (bed) of a crisp DL axiom τ w.r.t. $\mathcal{O}$ is defined as

$$\mathsf{bed}(\mathcal{O}, \tau) = \sup \{\alpha \mid \mathcal{O} \models \langle \tau, \alpha \rangle\} .$$

In the following, we will restrict to the case of SFL and Gödel fuzzy logics.

Before going on, we will add some useful remarks.

Remark 23. Please note that under SFL, if $\mathcal{O} \models \langle C \sqsubseteq D, \alpha \rangle$ then from $\alpha > 0$ it follows that for all models $\mathcal{I}$ of $\mathcal{O}$ and all $x \in \Delta^{\mathcal{I}}$ we have that $C^{\mathcal{I}}(x) \leq D^{\mathcal{I}}(x)$. Specifically, note that if D is $\bot$ then $C^{\mathcal{I}}(x) = 0$. Furthermore, this latter property is also true for Gödel logic and Product logic, but not for Łukasiewicz logic.

Remark 24. Like for Remark 5, fuzzy concept assertions $\langle C(a), \alpha \rangle$ and fuzzy role assertions $\langle r(a, b), \alpha \rangle$ can be represented in e.g. nominal safe fuzzy $\mathcal{ELO}_{\bot}^{+}$ via the mapping $\langle C(a), \alpha \rangle \mapsto \langle \{a\} \sqsubseteq C, \alpha \rangle$ and $\langle r(a, b), \alpha \rangle \mapsto \langle \{a\} \sqsubseteq \exists r.\{b\}, \alpha \rangle$ under Gödel, Łukasiewicz and Product logics, but not under SFL, as for instance $(\{a\} \sqsubseteq C)^{\mathcal{I}} \in \{0, 1\}$ holds, for any fuzzy interpretation $\mathcal{I}$ (see Remark 23).

Remark 25. Since the implication used in the semantics of the concept inclusions satisfies the ordering property, the conditions for domain and range restrictions are in agreement with Remark 17 in the sense that they are equivalent to $\exists r.\top \sqsubseteq A$ and $\top \sqsubseteq \forall r.A$, respectively.

Remark 26. Note that according to Table 4 the only difference between SFL and Gödel logic for fuzzy $\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ is related to the semantics of concept inclusion, role inclusion, and role composition axioms.

An important consequence of Remark 26 about an inclusion axiom $\langle \tau, \alpha \rangle$ under SFL is that $\tau^{\mathcal{I}} \in \{0, 1\}$ for all fuzzy interpretations $\mathcal{I}$ and, thus, under SFL all truth values in concept inclusion, role inclusion and role composition axioms can be replaced with the truth value 1. So, in the following, let $\mathbf{1}(\mathcal{O})$ be as the $\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ ontology $\mathcal{O}$ in which all concept inclusion, role inclusion and role composition axioms $\langle \tau, \alpha \rangle$ have been replaced with $\langle \tau, 1 \rangle$. Now, by Remark 26, as the semantics of fuzzy assertions is the same both for SFL as well as for Gödel logic, we have that

Proposition 27. *Let $\mathcal{O}$ be an $\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ ontology and ϕ a fuzzy $\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ axiom. Then the following statements are equivalent:*

1. *$\mathcal{O} \models \phi$ under SFL;*
2. *$\mathbf{1}(\mathcal{O}) \models \phi$ under SFL;*
3. *$\mathbf{1}(\mathcal{O}) \models \phi$ under Gödel logic.*

Corollary 28. *In agreement with Proposition 27, the following entailments are easily verified:*

1. *$\langle A(a), \alpha \rangle, \langle A \sqsubseteq B, \beta \rangle \models \langle B(a), \alpha \rangle$ under SFL;*
2. *$\langle A(a), \alpha \rangle, \langle A \sqsubseteq B, 1 \rangle \models \langle B(a), \alpha \rangle$ under SFL;*
3. *$\langle A(a), \alpha \rangle, \langle A \sqsubseteq B, 1 \rangle \models \langle B(a), \alpha \rangle$ under Gödel logic.*

The consequence of Proposition 27 is that fuzzy $\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ under SFL is a special case of fuzzy $\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ under Gödel logic. Hence, in the rest of this paper we are not going to consider SFL further.

To simplify the writing and reading, we will denote a fuzzy DL $\mathcal{L}$ under Gödel logic with G-$\mathcal{L}$, while denote with Ł-$\mathcal{L}$ the case Łukasiewicz logic is considered instead.

3.2. Reasoning in G-$\mathcal{EL}_{\bot}^{+}$ and nominal safe G-$\mathcal{ELO}_{\bot}^{+}$

At first, given the non-polynomial computational complexity results provided in the related work section (see, e.g. [17,20,21]), in what follows, to guarantee a polynomial time reasoning algorithm, we will stick to Gödel logics only.

$(FN_0^{\bot})$ $\frac{\langle C_1 \sqcap \bot \sqcap C_2 \sqsubseteq D, \alpha\rangle}{\langle \bot \sqsubseteq D, \alpha\rangle}$:

$(FN_0^{\top})$ $\frac{\langle C_1 \sqcap \top \sqcap C_2 \sqsubseteq D, \alpha\rangle}{\langle C_1 \sqcap C_2 \sqsubseteq D, \alpha\rangle}$:

(FN_1) $\frac{\langle C_1 \sqcap C \sqcap C_2 \sqsubseteq D, \alpha\rangle}{\langle C \sqsubseteq A, 1\rangle\ ,\ \langle C_1 \sqcap A \sqcap C_2 \sqsubseteq D, \alpha\rangle}$: $C \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, A new concept name

(FN_2) $\frac{\langle \exists r.C \sqsubseteq D, \alpha\rangle}{\langle C \sqsubseteq A, 1\rangle\ \langle \exists r.A \sqsubseteq D, \alpha\rangle}$: $C \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$, A new concept name

(FN_3) $\frac{\langle C \sqsubseteq D, \alpha\rangle}{\langle C \sqsubseteq A, 1\rangle\ \langle A \sqsubseteq D, \alpha\rangle}$: $C, D \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, A new concept name

(FN_4) $\frac{\langle B \sqsubseteq \exists r.C, \alpha\rangle}{\langle B \sqsubseteq \exists r.A, \alpha\rangle\ \langle A \sqsubseteq C, 1\rangle}$: $C \notin \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$, A new concept name

(FN_5) $\frac{\langle C \sqsubseteq D_1 \sqcap D_2, \alpha\rangle}{\langle C \sqsubseteq D_1, \alpha\rangle\ \langle C \sqsubseteq D_2, \alpha\rangle}$:

Fig. 4. Normal form transformation rules for G-$\mathcal{ELO}_{\bot}^{+}$.

(FR_0) $\frac{}{\langle A \sqsubseteq A, 1\rangle}$: $A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

(FR_1) $\frac{}{\langle C \sqsubseteq \top, 1\rangle}$: $C \sqsubseteq \top$ in normal form

(FR_2) $\frac{}{\phi}$: $\phi \in \mathcal{O}$

$(FR_{\sqsubseteq}^{1})$ $\frac{\langle C_1 \sqsubseteq C_2, \alpha_1\rangle\ \langle C_2 \sqsubseteq C_3, \alpha_2\rangle}{\langle C_1 \sqsubseteq C_3, \alpha_1 \otimes \alpha_2\rangle}$:

$(FR_{\sqsubseteq}^{2})$ $\frac{\langle A \sqsubseteq B_1, \alpha_1\rangle ... \langle A \sqsubseteq B_n, \alpha_n\rangle\ \langle B_1 \sqcap ... \sqcap B_n \sqsubseteq B, \beta\rangle}{\langle A \sqsubseteq B, \alpha_1 \otimes ... \otimes \alpha_n \otimes \beta\rangle}$: $A, B, B_i \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

$(FR_{\exists})$ $\frac{\langle A \sqsubseteq \exists r.B, \alpha_1\rangle\ \langle B \sqsubseteq C, \alpha_2\rangle}{\langle A \sqsubseteq \exists r.C, \alpha_1 \otimes \alpha_2\rangle}$: $A, B, C, \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $C \neq \bot$

$(FR_{\bot})$ $\frac{\langle A \sqsubseteq \exists r.B, \alpha_1\rangle\ \langle B \sqsubseteq \bot, \alpha_2\rangle}{\langle A \sqsubseteq \bot, \alpha_1 \otimes \alpha_2\rangle}$: $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

(FR_{H}) $\frac{\langle A \sqsubseteq \exists r.B, \alpha_1\rangle\ \langle r \sqsubseteq s, \alpha_2\rangle}{\langle A \sqsubseteq \exists s.B, \alpha_1 \otimes \alpha_2\rangle}$: $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

$(FR_{H^{\star}})$ $\frac{\langle r_1 \sqsubseteq r_2, \alpha_1\rangle\ \langle r_2 \sqsubseteq r_3, \alpha_2\rangle}{\langle r_1 \sqsubseteq r_3, \alpha_1 \otimes \alpha_2\rangle}$:

$(FR_{\circ})$ $\frac{\langle A_1 \sqsubseteq \exists r_1.B, \alpha_1\ \langle B \sqsubseteq \exists r_2.A_2, \alpha_2\rangle\rangle\ \langle r_1 \circ r_2 \sqsubseteq s, \alpha_3\rangle}{\langle A_1 \sqsubseteq \exists s.A_2, \alpha_1 \otimes \alpha_2 \otimes \alpha_3\rangle}$: $A_1, B, A_2 \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$

Fig. 5. Inference rules for G-$\mathcal{EL}_{\bot}^{+}$ ontologies in normal form.

Like for the crisp case, we start with the inference rules for the case of assertion free G-$\mathcal{EL}_{\bot}^{+}$ and then, will show how to easily extend it to cover nominal safe G-$\mathcal{ELO}_{\bot}^{+}$ as well. At first, we focus here on the subsumption problem in G-$\mathcal{EL}_{\bot}^{+}$, *i.e.*, on the problem to decide whether $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$, where $\mathcal{O}$ is an assertion free G-$\mathcal{EL}_{\bot}^{+}$ ontology and $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$.[7]

In the following, we are going to parallel the inference rules we have presented for crisp assertion free $\mathcal{EL}_{\bot}^{+}$. We assume that whenever we have inferred $\langle \tau, \alpha\rangle$ and $\langle \tau, \beta\rangle$, the two axioms are replaced with $\langle \tau, \max(\alpha, \beta)\rangle$. In this way, for each τ, we have at most one fuzzy axiom involving it.

To start with, Fig. 4 shows the inference rules to transform a G-$\mathcal{ELO}_{\bot}^{+}$ ontology, and thus also $\mathcal{O}$, into normal form. The inference rules over a G-$\mathcal{EL}_{\bot}^{+}$ ontology in normal form are depicted in Fig. 5 instead.

In G-$\mathcal{EL}_{\bot}^{+}$, given a fuzzy ontology $\mathcal{O}$, let $\mathsf{fclosure}(\mathcal{O})$ be a set that is closed under the rules in Figs. 4 and 5 applied to $\mathcal{O}$. Note that indeed for $\langle C \sqsubseteq D, \alpha\rangle \in \mathsf{fclosure}(\mathcal{O})$ we always have $\alpha > 0$.

[7] Note that like for the crisp case, $\mathcal{O} \models \langle C \sqsubseteq D, \alpha\rangle$ iff $\mathcal{O} \cup \{A \sqsubseteq C, D \sqsubseteq B\} \models \langle A \sqsubseteq B, \alpha\rangle$, where A, B are new atomic concepts.

Example 29. Given the fuzzy ontology $\mathcal{O}_3$ defined in Example 21, the normal form of $N(\mathcal{O}_3)$ is

$$\{\langle A \sqsubseteq \exists s.C, 0.8\rangle, \langle A \sqsubseteq \top, 0.8\rangle, \langle \exists s.N_a, 0.8\rangle, \langle C \sqsubseteq B, 0.7\rangle, \langle \exists s.B \sqsubseteq B, 0.6\rangle\} .$$

It is easy to see that rule $FR_{\exists}$ adds $\langle A \sqsubseteq \exists s.B, 0.7\rangle$ to $\mathsf{fclosure}(N(\mathcal{O}_3))$, while rule $FR^1_{\sqsubseteq}$ adds $\langle A \sqsubseteq B, 0.6\rangle$.

The following proposition, which is the fuzzy variant of Proposition 11, can be shown (proof in Appendix A).

Proposition 30. *In G-$\mathcal{EL}^+_\bot$, let $\mathcal{O}$ be an assertion free fuzzy ontology and let $A, B \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}$. Then,*

1. $\mathsf{fclosure}(\mathcal{O})$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.
2. $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$ *iff there is* $\langle A \sqsubseteq B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ *with* $\beta \geq \alpha$, *or* $\langle A \sqsubseteq \bot, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$.[8]

Now, we present the fuzzy variants of Propositions 6 and 7. Like for the crisp case, we define $N(x)$ to be the result of replacing each occurrence of each nominal $\{a\}$ in expression x with N_a. Then the following fuzzy variants of Propositions 6 and 7 can be shown (proofs in Appendix A).

Proposition 31. *In G-$\mathcal{ELO}^+_\bot$, let $\mathcal{O}$ be a fuzzy ontology and ϕ a fuzzy axiom that do not contain atomic concepts of the form N_a. Then,*

1. *if* $N(\mathcal{O}) \models N(\phi)$ *then* $\mathcal{O} \models \phi$;
2. *if* $N(\mathcal{O}) \models N_a \sqsubseteq \bot$ *for some* a *then* $\mathcal{O}$ *is inconsistent.*

Proposition 32. *In G-$\mathcal{ELO}^+_\bot$, let $\mathcal{O}$ be a nominal safe fuzzy ontology and ϕ a safe fuzzy GCI that do not contain atomic concepts of the form N_a. Assume that $N(\mathcal{O}) \not\models N_a \sqsubseteq \bot$ for all a. Then,*

1. $\mathcal{O}$ *is consistent;*
2. *if* $\mathcal{O} \models \phi$ *then* $N(\mathcal{O}) \models N(\phi)$.

By Propositions 31 and 32, and by Remark 24, we get easily

Corollary 33. *In G-$\mathcal{ELO}^+_\bot$, let $\mathcal{O}$ be a nominal safe fuzzy ontology and let $A, B \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}$. Then,*

1. $\mathsf{fclosure}(N(\mathcal{O}))$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.
2. $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$ *iff there is* $\langle N(A) \sqsubseteq N(B), \beta\rangle \in \mathsf{fclosure}(N(\mathcal{O}))$ *with* $\beta \geq \alpha$, *or* $\langle N(A) \sqsubseteq \bot, \beta\rangle \in \mathsf{fclosure}(N(\mathcal{O}))$.

Remark 34. Note that if we restrict our attention to fuzzy concept subsumption then rule $FR_{H^\star}$ is in fact not needed.

3.3. Reasoning with domain, range and reflexive role restrictions

We next address the case of fuzzy reflexive role, range and domain restrictions. We recap that for such axioms, we postulated that the involved degree of truth is always 1. We make the following fuzzy analogue restrictions of Section 2.2.2: for ontology $\mathcal{O}$, roles r, s, we write $\mathcal{O} \vdash \langle r \sqsubseteq s, \alpha\rangle$ iff $r = s$ (in this case $\alpha = 1$); or $\mathcal{O}$ contains role inclusions

$$\langle r_1 \sqsubseteq r_2, \alpha_1\rangle, \ldots, \langle r_{n-1} \sqsubseteq r_n, \alpha_{n-1}\rangle \text{ with } r = r_1 \text{ and } s = r_n \text{ with } \alpha = \alpha_1 \otimes \ldots \otimes \alpha_{n-1} .$$

Note that $\mathcal{O} \vdash \langle r \sqsubseteq s, \alpha\rangle$ implies $\alpha > 0$. Furthermore, we write $\mathcal{O} \vdash \mathsf{ran}(r) \sqsubseteq A$ if there is a role s with $\mathcal{O} \vdash \langle r \sqsubseteq s, \alpha\rangle$ and $\mathsf{ran}(s) \sqsubseteq A \in \mathcal{O}$.

[8] See Remark 23 for the case $\langle A \sqsubseteq \bot, \beta\rangle$.

Now, the mentioned restriction is as follows:

($\star_f$) If $\langle r_1 \circ \ldots \circ r_n \sqsubseteq s, \alpha\rangle \in \mathcal{O}$ with $1 \leq n \leq 2$ and $\mathcal{O} \vdash \mathsf{ran}(s) \sqsubseteq A$, then $\mathcal{O} \vdash \mathsf{ran}(r_n) \sqsubseteq A$.

Remark 35. Like for Remark 15, the condition ($\star_f$) is true if the fuzzy role inclusion is a reflexive role restriction, a role hierarchy statement or a transitivity statement.

Remark 36. Similarly to Remark 16, we postulate that Fuzzy OWL 2 EL, which is the restriction of Fuzzy OWL 2 [13] to the OWL 2 EL sublanguage, needs to satisfy the above-mentioned condition ($\star_f$). As discussed in Remark 1 for the crisp case, fuzzy $\mathcal{ELO}^+_\bot(\mathbf{D})$ with domain, range and reflexive role restrictions is essentially the same as Fuzzy OWL 2 EL.

Reasoning We next show how to deal with these additional axioms in order to decide fuzzy concept subsumption. At first, by Remark 17 $\mathsf{dom}(r) \sqsubseteq A$ can be replaced, *w.l.o.g.*, with a concept inclusion axiom $\langle \exists r.\top \sqsubseteq A, 1\rangle$, and, thus, we have not to deal with domain restrictions further. In order to support reflexive role axioms, it is enough to adapt rules FR_H, $FR_{H^\star}$, and $FR_\circ$ to consider the case $\epsilon \sqsubseteq r$ as well. We next parallel the procedure for crisp $\mathcal{ELO}^+_\bot$ to remove range restriction axioms. Let us assume that $\mathcal{O}$ is in normal form. We proceed as follows.

RangeElimination Step:

1. For each role r, let $\mathsf{ran}_{\mathcal{O}}(r) = \{A \mid \mathcal{O} \vdash \mathsf{ran}(r) \sqsubseteq A\}$.
2. For each $\langle C \sqsubseteq \exists r.B, \alpha\rangle \in \mathcal{O}$, introduce a new atomic concept $X_{r,B}$.
3. Let $\mathcal{O}'$ be obtained from $\mathcal{O}$ by removing all range restriction axioms and performing additionally the following actions:
   (a) exchange every $\langle C \sqsubseteq \exists r.B, \alpha\rangle$ with the axioms $\langle C \sqsubseteq \exists r.X_{r,B}, \alpha\rangle$, $\langle X_{r,B} \sqsubseteq B, 1\rangle$, and $\langle X_{r,B} \sqsubseteq A, 1\rangle$ for all $A \in \mathsf{ran}_{\mathcal{O}}(r)$;
   (b) if $\epsilon \sqsubseteq r \in \mathcal{O}$, then add $\top \sqsubseteq A$ for all $A \in \mathsf{ran}_{\mathcal{O}}(r)$.

As for the crisp case, the size of $\mathcal{O}'$ is quadratically bounded by $|\mathcal{O}|$. The following is the fuzzy analogue of Proposition 18 (proof in Appendix A).

Proposition 37. *In G-$\mathcal{EL}^+_\bot$, let $\mathcal{O}$ be an assertion free fuzzy ontology, which may contain range and reflexive role restriction axioms, and $A, B \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}$. Then, $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$ iff $\mathcal{O}' \models \langle A \sqsubseteq B, \alpha\rangle$, where $\mathcal{O}'$ has been determined by the* **RangeElimination** *step.*

Of course, by Propositions 31 and 32, and by Remark 24 we have that

Corollary 38. *Proposition 37 holds also for nominal safe G-$\mathcal{ELO}^+_\bot$ ontologies, which may contain range and reflexive role restriction axioms.*

Therefore, by Proposition 33

Corollary 39. *In G-$\mathcal{ELO}^+_\bot$, subsumption can be determined in polynomial time for nominal safe fuzzy ontologies, even in the presence of domain, range and reflexive role restriction axioms.*

Eventually, we address the best entailment degree problem. From [59] we know that $\mathsf{bed}(\mathcal{O}, A \sqsubseteq B)$ has to be one of the truth values that occur in $\mathcal{O}$ and, thus, a binary search over these values has been proposed to solve the best entailment degree problem in general. However, like in [49], for the specific case of nominal safe G-$\mathcal{ELO}^+_\bot$ ontologies we can do better: indeed, by analyzing on how $\mathsf{fclosure}$ is determined it turns out that the axioms in it are already "maximized" in their truth and, thus, we immediately have the following:

Proposition 40. *In G-$\mathcal{ELO}^+_\bot$, let $\mathcal{O}$ be a nominal safe fuzzy ontology, which may contain domain, range and reflexive role restriction axioms, and let $A, B \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}$. Then,*

$$(\mathbf{D}_1)\ \frac{A \sqsubseteq \exists t.\mathbf{d}_1, \ldots, A \sqsubseteq \exists t.\mathbf{d}_n}{A \sqsubseteq \bot} : \begin{array}{l} A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}, a \text{ new individual}, \textit{and} \\ \{(\exists t.\mathbf{d}_1)(a), \ldots, (\exists t.\mathbf{d}_n)(a)\} \text{ unsatisfiable} \end{array}$$

$$(\mathbf{D}_2)\ \frac{A \sqsubseteq \exists t.\mathbf{d}_1, \ldots, A \sqsubseteq \exists t.\mathbf{d}_n}{A \sqsubseteq \exists t.\mathbf{d}} : \begin{array}{l} \exists t.\mathbf{d} \text{ occurs in } \mathcal{O}, \\ A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}, a \text{ new individual}, \textit{and} \\ \{(\exists t.\mathbf{d}_1)(a), \ldots, (\exists t.\mathbf{d}_n)(a)\} \models (\exists t.\mathbf{d})(a) \end{array}$$

Fig. 6. Inference rules for crisp concrete domains.

1. *bed*$(\mathcal{O}, A \sqsubseteq B) = \alpha$ *iff* $\alpha = \max(\beta_1, \beta_2)$ *with*
   (a) *if* $\langle A \sqsubseteq B, \gamma_1 \rangle \in \mathsf{fclosure}(\mathcal{O})$ *then* $\beta_1 = \gamma_1$ *else* $\beta_1 = 0$*;*
   (b) *if* $\langle A \sqsubseteq \bot, \gamma_2 \rangle \in \mathsf{fclosure}(\mathcal{O})$*, then* $\beta_2 = 1$ *else* $\beta_2 = 0$.
2. *bed*$(\mathcal{O}, A \sqsubseteq B)$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.

3.4. Reasoning with fuzzy concrete domains

To start with, it is easily verified that the inference rules in Fig. 4 also apply to G-$\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ in the sense that they transform a G-$\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ ontology into normal form. Therefore, in the following, we always assume that ontologies are in normal form. Next, for ease of exposition, we recap the case of crisp concrete domains before addressing fuzzy ones.

P-admissible crisp concrete domains For the moment, let us restrict data type predicates to $\{\geq_v, \leq_v, =_v\}$. As a consequence, concept expressions of the form $\exists t.\mathbf{d}$ are always crisp: *i.e.*, for all interpretations $\mathcal{I}$, for all $x \in \Delta^{\mathcal{I}}$, we have that $(\exists t.\mathbf{d})^{\mathcal{I}}(x) \in \{0, 1\}$ (recall that we are assuming that t is crisp). Furthermore, let us assume that the ontology $\mathcal{O}$ is crisp as well. Then, Fig. 6 illustrates the inference rules for crisp concrete domains that are an adaption of the rules **CR7** and **CR8** in [1] to our setting.

Remark 41. Please note that in rules $(\mathbf{D}_1)$ and $(\mathbf{D}_2)$, given an atomic concept A and datatype property t, the axioms $A \sqsubseteq \exists t.\mathbf{d}_i$ are all those in the ontology or inferred so far. Note also that in rule $(\mathbf{D}_1)$, testing the (un)satisfiability of $\{(\exists t.\mathbf{d}_1)(a), \ldots, (\exists t.\mathbf{d}_n)(a)\}$ is the same as to test whether the set of constraints

$$\mathbf{d}_i(t) = 1 \ (1 \leq i \leq n)$$

over rational-valued variable t and functions d_i has a solution or not.[9]

On the other hand, concerning $(\mathbf{D}_2)$, the condition

$$\{(\exists t.\mathbf{d}_1)(a), \ldots, (\exists t.\mathbf{d}_n)(a)\} \models (\exists t.\mathbf{d})(a)$$

equals to test whether the optimization problem

$$\begin{array}{ll} \min & \mathbf{d}(t) \\ & s.t.\ \mathbf{d}_i(t) = 1 \ (1 \leq i \leq n) \end{array}$$

has solution 1.

As [1] shows, there are however some constraints on the fuzzy concrete domain that need to be met to guarantee that these two rules are correct and the procedure runs in polynomial time: this is called *p-admissibility*. The adaption of the notion of p-admissibility as by [1] to our setting is as follows.

Definition 42 *(P-admissible concrete domain).* A concrete domain is *p-admissible* if the following conditions hold:

1. satisfiability and entailment decision problems indicated in rules $(\mathbf{D}_1)$ and $(\mathbf{D}_2)$ are decidable in polynomial time; and

[9] Note that this reduction holds as the datatype property t in $\exists t.\mathbf{d}_i$ is functional.

2. *convexity*: if

$$\bigcup_{1\leq i\leq n} \{(\exists t.\mathbf{d}_i)(a)\} \models (\exists t.\mathbf{d}'_1 \sqcup \ldots \sqcup \exists t.\mathbf{d}'_m)(a)$$

then

$$\bigcup_{1\leq i\leq n} \{(\exists t.\mathbf{d}_i)(a)\} \models (\exists t.\mathbf{d}'_j)(a)$$

for some $1 \leq j \leq m$.

Before moving to the fuzzy case, it is worth mentioning some remarks.

Remark 43. Note that the main difference to [1] is that convexity does not require to involve various datatype properties, but we may stick to a datatype property t only. That is, a direct translation of p-admissibility according to [1] would be: if

$$\bigcup_{1\leq i\leq n} \{(\exists t_i.\mathbf{d}_i)(a)\} \models (\exists t'_1.\mathbf{d}'_1 \sqcup \ldots \sqcup \exists t'_m.\mathbf{d}'_m)(a)$$

then

$$\bigcup_{1\leq i\leq n} \{(\exists t_i.\mathbf{d}_i)(a)\} \models (\exists t'_j.\mathbf{d}'_j)(a)$$

for some $1 \leq j \leq m$. In our setting we may replace the t_i and t'_j with t.

The reason is due to the fact that in [1] (and also in [49]) fuzzy concrete domain predicates may be n-ary ($n \geq 1$) and, thus, may involve various datatype properties, while in our setting (and that of Fuzzy OWL 2) concrete domain predicates are unary only.

Remark 44. Note that the first condition on p-admissibility corresponds to requiring that the optimization decision problems in Remark 41 are decidable in polynomial time. On the other hand, the second condition of p-admissibility can be reduced to optimization decision problems as follows. Assume that the optimization problem

$$\begin{array}{ll} \min & z \\ s.t. & \mathbf{d}'_j(t) \leq z \ (1 \leq j \leq m) \\ & \mathbf{d}_i(t) = 1 \ (1 \leq i \leq n) \end{array}$$

has solution 1. Then, for some $1 \leq j \leq m$, the optimization problem

$$\begin{array}{ll} \min & \mathbf{d}'_j(t) \\ s.t. & \mathbf{d}_i(t) = 1 \ (1 \leq i \leq n) \end{array}$$

has solution 1 as well.

Remark 45. [1] shows that $\{\geq_v, =_v\}$ and $\{\leq_v, =_v\}$ are both p-admissible, while e.g. $\{\leq_v, >_v\}$ is not. In the latter case note that

$$(\exists t. \leq_{10})(a) \models ((\exists t. \leq_5) \sqcup (\exists t. >_5))(a),$$

while

$$(\exists t. \leq_{10})(a) \not\models (\exists t. \leq_5)(a)$$
$$(\exists t. \leq_{10})(a) \not\models (\exists t. >_5)(a)$$

$$(F\mathbf{D}_1)\ \frac{\langle A \sqsubseteq \exists t.\mathbf{d}_1, \alpha_1\rangle, \ldots, \langle A \sqsubseteq \exists t.\mathbf{d}_n, \alpha_n\rangle}{A \sqsubseteq \bot} : \begin{array}{l} A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}, a \text{ new individual }, \textit{and} \\ \{\langle(\exists t.\mathbf{d}_1)(a), \alpha_1\rangle, \ldots, \\ \quad \langle(\exists t.\mathbf{d}_n)(a), \alpha_n\rangle\} \text{ unsatisfiable} \end{array}$$

$$(F\mathbf{D}_2)\ \frac{\langle A \sqsubseteq \exists t.\mathbf{d}_1, \alpha_1\rangle, \ldots, \langle A \sqsubseteq \exists t.\mathbf{d}_n, \alpha_n\rangle}{\langle A \sqsubseteq \exists t.\mathbf{d}, \beta\rangle} : \begin{array}{l} A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}, \exists t.\mathbf{d} \text{ occurs in } \mathcal{O}, \\ a \text{ new individual }, \textit{and} \\ 0 < \beta = \mathsf{bed}(\cup_{1 \le i \le n}\{\langle(\exists t.\mathbf{d}_i)(a), \alpha_i\rangle\}, \exists t.\mathbf{d}) \end{array}$$

Fig. 7. Inference rules for fuzzy concrete domains.

Furthermore, [1] shows that the failure of the convexity property of $\mathbf{D}$ implies that one may simulate concept disjunction on the right-hand side of GCIs. In fact, e.g. given $\mathcal{O}$ with axioms

$$\begin{aligned} A &\sqsubseteq \exists t. \le_{10} \\ \exists t. \le_5 &\sqsubseteq B_1 \\ \exists t. >_5 &\sqsubseteq B_2 \\ B_1 &\sqsubseteq B \\ B_2 &\sqsubseteq B \end{aligned}$$

we have

$$\begin{aligned} &\mathcal{O} \models A \sqsubseteq B_1 \sqcup B_2 \\ &\mathcal{O} \models A \sqsubseteq B \\ &\mathcal{O} \not\models A \sqsubseteq B_i \ (i = 1, 2) \ . \end{aligned}$$

Therefore, the subsumption problem in $\mathcal{EL}(\mathbf{D})$ becomes as hard as subsumption in $\mathcal{ELU}$, which is a EXPTIME-complete problem. Therefore, for non-convex concrete domains $\mathbf{D}$, the subsumption problem in $\mathcal{EL}(\mathbf{D})$ is EXPTIME-hard and, thus, the inference rules in Fig. 6 cannot be sound and complete in this case.

Fuzzy p-admissible concrete domains For G-$\mathcal{EL}^{++}$, [49] provides similar rules as **CR7** and **CR8** in [1]. In our setting, the inference rules for fuzzy concrete domains are instead illustrated in Fig. 7 and are an adaption of those in Fig. 6 to the fuzzy case.

Remark 46. Similarly to the crisp case (see Remark 41), in rule $(F\mathbf{D}_1)$, testing the (un)satisfiability of $\{\langle(\exists t.\mathbf{d}_1)(a), \alpha_1\rangle, \ldots, \langle(\exists t.\mathbf{d}_n)(a), \alpha_n\rangle\}$ is the same as to test whether the set of constraints

$$\mathbf{d}_i(t) \ge \alpha_i \ (1 \le i \le n)$$

over rational-valued variable t and functions d_i has a solution or not.

On the other hand, concerning $(F\mathbf{D}_2)$, determining

$$\mathsf{bed}(\cup_{1 \le i \le n}\{\langle(\exists t.\mathbf{d}_i)(a), \alpha_i\rangle\}, \exists t.\mathbf{d})$$

equals to solve the optimization problem

$$\begin{aligned} \min \quad & \mathbf{d}(t) \\ s.t. \ & \mathbf{d}_i(t) \ge \alpha_i \ (1 \le i \le n) \ . \end{aligned}$$

Of course, to restrict the computation to polynomial time and to guarantee completeness, we need to extend the notion of crisp p-admissibility to the fuzzy case as well (see also [49]). Essentially, we need to guarantee that the rules $(F\mathbf{D}_1)$ and $(F\mathbf{D}_2)$ are computable in polynomial time and that the set of concrete fuzzy predicates is convex.

Definition 47 *(Fuzzy p-admissible concrete domain).* A fuzzy concrete domain is *fuzzy p-admissible* if the following conditions hold:

1. satisfiability and best entailment problems indicated in rules $(F\mathbf{D}_1)$ and $(F\mathbf{D}_2)$ can be solved in polynomial time; and
2. *convexity*: if

$$\beta = \mathsf{bed}(\cup_{1\leq i\leq n}\{\langle(\exists t.\mathbf{d}_i)(a), \alpha_i\rangle\}, (\exists t.\mathbf{d}'_1 \sqcup \ldots \sqcup \exists t.\mathbf{d}'_m)(a)) > 0$$

then

$$\beta = \mathsf{bed}(\cup_{1\leq i\leq n}\{\langle(\exists t.\mathbf{d}_i)(a), \alpha_i\rangle\}, (\exists t.\mathbf{d}'_j)(a))$$

for some $1 \leq j \leq m$.

Let us now add a couple of remarks.

Remark 48. Similarly as Remark 44, note that the first condition on fuzzy p-admissibility corresponds to requiring that the optimization decision problems in Remark 46 are decidable in polynomial time. On the other hand, the second condition of p-admissibility can be reduced to optimization decision problems as follows. Assume that the optimization problem

$$\begin{array}{ll} \min & z \\ s.t. & \mathbf{d}'_j(t) \leq z \ (1 \leq j \leq m) \\ & \mathbf{d}_i(t) \geq \alpha_i \ (1 \leq i \leq n) \end{array}$$

has solution $\beta > 0$. Then, for some $1 \leq j \leq m$, the optimization problem

$$\begin{array}{ll} \min & \mathbf{d}'_j(t) \\ s.t. & \mathbf{d}_i(t) \geq \alpha_i \ (1 \leq i \leq n) \end{array}$$

has solution β as well.

Remark 49. [49] shows that $\{rs(q-10, q) \mid q \in \mathbb{Q}\}$ and $\{ls(q, q+10) \mid q \in \mathbb{Q}\}$ are both fuzzy p-admissible. Note that we may further add $\geq_v$ to the former and $\leq_v$ to the latter, respectively, without losing fuzzy p-admissibility.

Note also that there can be non-convex fuzzy concrete domains using convex fuzzy sets, e.g. $\{tri(q-1, q, q+1) \mid q \in \mathbb{Q}\}$ and $\{trz(q-1, q, q+1, q+2) \mid q \in \mathbb{Q}\}$ are not convex and, thus, not fuzzy p-admissible, as the following example illustrates. For convenience, let

$$\mathbf{d}_1 := tri(0, 1, 2)$$
$$\mathbf{d}_2 := tri(-0.5, 0.5, 1.5)$$
$$\mathbf{d}_3 := tri(0.5, 1.5, 2.5)\ .$$

Then, it can be verified that

$$0.5 = \mathsf{bed}(\langle(\exists t.\mathbf{d}_1)(a), 0.5\rangle, (\exists t.\mathbf{d}_2 \sqcup (\exists t.\mathbf{d}_3)(a))$$

while

$$0 = \mathsf{bed}(\langle(\exists t.\mathbf{d}_1)(a), 0.5\rangle, (\exists t.\mathbf{d}_2)(a))$$
$$0 = \mathsf{bed}(\langle(\exists t.\mathbf{d}_1)(a), 0.5\rangle, (\exists t.\mathbf{d}_3)(a))$$

and, thus, $\{tri(q-1, q, q+1) \mid q \in \mathbb{Q}\}$ is not convex. Non convexity introduces disjunction: in fact, consider the ontology using the above triangular functions

$$\begin{array}{l} \mathcal{O} = \{\ \langle\{a\} \sqsubseteq \exists t.\mathbf{d}_1, 0.5\rangle, \\ \qquad \exists t.\mathbf{d}_2 \sqsubseteq C, \\ \qquad \exists t.\mathbf{d}_3 \sqsubseteq D, \\ \qquad C \sqsubseteq A, \\ \qquad D \sqsubseteq A\ \}\ . \end{array}$$

Then, it can be verified that

$$\mathcal{O} \models \langle \{a\} \sqsubseteq C \sqcup D, 0.5 \rangle$$
$$\mathcal{O} \models \langle \{a\} \sqsubseteq A, 0.5 \rangle$$
$$\mathcal{O} \not\models \langle \{a\} \sqsubseteq C, \alpha \rangle \ (\alpha > 0)$$
$$\mathcal{O} \not\models \langle \{a\} \sqsubseteq D, \alpha \rangle \ (\alpha > 0) \ .$$

Therefore, disjunction on the right-hand side of GCIs has been introduced and, thus, the rules in Fig. 7 cannot be complete (see Remark 45) for nominal safe G-$\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ if fuzzy p-admissibility is not satisfied.

Moreover, as shown in [49], in the fuzzy case there is yet another problem that has to be addressed.

Example 50. [49, Example 11] shows that fuzzy concrete domains may affect running time and truth set. Specifically, unlike nominal safe G-$\mathcal{ELO}$, in which the set of truth values in $\mathsf{fclosure}(\mathcal{O})$ is the set of truth values occurring in an ontology, this may no longer be true in presence of fuzzy p-admissible concrete domains.

In fact, consider the following nominal safe ontology $\mathcal{O}$ with axioms[10]

$$\langle \{a\} \sqsubseteq \exists t.rs(10, 20), 0.5 \rangle$$
$$\exists t.rs(10, 20) \sqsubseteq A$$
$$A \sqsubseteq \exists t.rs(10.5, 20.5) \ .$$

For $\mathcal{O}$ it is not difficult to see that, by applying the (incomplete) inference rule set in [49], the degree of truth of $A(a)$ increases each step by 0.05 until it reaches 1, *i.e.*, $\mathcal{O} \models \langle A(a), 1 \rangle$ holds. Therefore, polynomial execution time w.r.t. $|\mathcal{O}|$ is not guaranteed.

Remark 51. Any crisp p-admissible concrete domain does not suffer of the problem risen in Example 50.

To address the issue shown in Example 50, [49] proposes two solutions, which we adapt to the Fuzzy OWL 2 EL context, namely (i) restricting the usage of fuzzy concrete domains; or (ii) using *strict* fuzzy p-admissible concrete domains.

Definition 52 *(Loose fuzzy p-admissible concrete domain).* A fuzzy p-admissible concrete domain is *loose* if it satisfies the following DL language restriction:

- an expression $\exists t.\mathbf{d}$ may only occur in the form

  $$\langle (\exists t.\mathbf{d})(a), \alpha \rangle, \ \langle \{a\} \sqsubseteq \exists t.\mathbf{d}, \alpha \rangle, \ \langle \exists t.\mathbf{d} \sqsubseteq C, \alpha \rangle \ ,$$

  where C can be some arbitrary concept description as long as neither it, nor any of its subconcepts, is an expression of the form $\exists t'.\mathbf{d}'$.

Definition 53 *(Strict fuzzy p-admissible concrete domain).* A fuzzy p-admissible concrete domain is *strict* if it satisfies the following conditions: for all

$$\mathcal{O} = \bigcup_{1 \le i \le n} \{\langle (\exists t.\mathbf{d}_i)(a), \alpha_i \rangle\};$$

for all $\exists t.\mathbf{d}'$ with

$$\alpha = \mathsf{bed}(\mathcal{O}, (\exists t.\mathbf{d}')(a));$$

for all $\beta \in [0, 1]$ with

[10] We corrected the example as the original one seems not correct.

$$\mathcal{O}' = \bigcup_{1 \leq i \leq n} \{\langle (\exists t.\mathbf{d}_i)(a), \min(\alpha_i, \beta)\rangle\};$$

for all

$$\mathcal{O}'' = \{\langle (\exists t.\mathbf{d}'')(a), \alpha''\rangle \mid \langle (\exists t.\mathbf{d}'')(a), \alpha''\rangle \in \mathcal{O} \text{ and } \alpha'' \geq \alpha\}$$

it holds that

1. $\min(\alpha, \beta) = \mathsf{bed}(\mathcal{O}', (\exists t.\mathbf{d}')(a))$;
2. if $\mathcal{O}$ is inconsistent, then $\mathcal{O}'$ is also inconsistent;
3. α occurs in $\mathcal{O}$;
4. $\alpha = \mathsf{bed}(\mathcal{O}'', (\exists t.\mathbf{d}')(a))$.

Essentially, the aim of the conditions of strict fuzzy p-admissibility is to avoid the creation of new truth degrees. Of course, by Remark 51 any crisp concrete domain, such as $\{\geq_v, =_v\}$ and $\{\leq_v, =_v\}$, is strict fuzzy p-admissible. [49] also provides a fuzzy variant $=_v^{\mathbb{Q}}$ of $=_v$ that is strict, which is defined over $\mathbb{Q} \times [0, 1]$:

$$=_v^{\mathbb{Q}} (v', \alpha) = \begin{cases} \alpha & \text{if } v' = v \\ 0 & \text{otherwise.} \end{cases}$$

While checking whether a concrete domain is loose is easily verified, this seems not immediate for the strict case. Nevertheless, we provide next two additional useful alternatives, in order to increase the choice of fuzzy p-admissible concrete domains in practice.

Definition 54 *(Finitely-valued fuzzy p-admissible concrete domain).* A fuzzy p-admissible concrete domain $\mathbf{D}$ is *finitely-valued* if each fuzzy concrete predicate $\mathbf{d}$ in $\mathbf{D}$ has as range the well-known set of truth degrees $L_{n_\mathbf{d}}$ ($n_\mathbf{d} \geq 2$), where

$$L_{n_\mathbf{d}} = \{0, \frac{1}{n_\mathbf{d} - 1}, \ldots, \frac{n_\mathbf{d} - 2}{n_\mathbf{d} - 1}, 1\} .$$

Please note that only concrete domain predicates are finitely-valued, while concepts and roles need not to be. Clearly, in case of a finitely-valued fuzzy p-admissible concrete domain $\mathbf{D}$, no new truth degree may be created beyond those appearing in a fuzzy ontology $\mathcal{O}$ and in

$$L_\mathbf{D} = \bigcup_{\mathbf{d} \text{ occurs in } \mathbf{D}} L_{n_\mathbf{d}} .$$

If we consider the size of $\mathbf{D}$ as a fixed parameter, so is $|L_\mathbf{D}|$, which allows us to compute $\mathsf{fclosure}(\mathcal{O})$ in polynomial time w.r.t. $|\mathcal{O}|$.

Next, we propose to define another type of p-admissible fuzzy concrete domain by adding some constraints on the occurrence of datatypes in an ontology to avoid circular usage of a datatype property, so that no new truth degrees are introduced. To this end, at first, we define the notion for G-$\mathcal{EL}_\perp^+(\mathbf{D})$ and then extend it to G-$\mathcal{ELO}_\perp^+(\mathbf{D})$.

So, consider an assertion free G-$\mathcal{EL}_\perp^+(\mathbf{D})$ ontology $\mathcal{O}$. We define next the relation *uses*, denoted $\rightsquigarrow_\mathcal{O}$, as follows. Consider $\langle C \sqsubseteq D, \alpha\rangle \in \mathcal{O}$

1. if atomic concept A occurs in C and atomic concept B occurs in D then A *uses* B, denoted $A \rightsquigarrow_\mathcal{O} B$;
2. if atomic concept A occurs in C and datatype restriction $\exists t.\mathbf{d}$ occurs in D then A *uses* $t_\mathbf{d}$, denoted $A \rightsquigarrow_\mathcal{O} t_\mathbf{d}$;
3. if datatype restriction $\exists t.\mathbf{d}$ occurs in C and atomic concept B occurs in D then $t_\mathbf{d}$ *uses* B, denoted $t_\mathbf{d} \rightsquigarrow_\mathcal{O} B$;
4. if datatype restriction $\exists t.\mathbf{d}$ occurs in C and datatype restriction $\exists t.\mathbf{d}'$ occurs in D then $t_\mathbf{d}$ *uses* $t_{\mathbf{d}'}$, denoted $t_\mathbf{d} \rightsquigarrow_\mathcal{O} t_{\mathbf{d}'}$;
5. if $x \rightsquigarrow_\mathcal{O} t_\mathbf{d}$ and $t_{\mathbf{d}'} \rightsquigarrow_\mathcal{O} y$, $\mathbf{d} \neq \mathbf{d}'$ and $\beta = \mathsf{bsd}(\emptyset, \exists t.\mathbf{d} \sqcap \exists t.\mathbf{d}') > 0$ then $t_\mathbf{d}$ *uses* $t_{\mathbf{d}'}$, denoted $t_\mathbf{d} \rightsquigarrow_\mathcal{O} t_{\mathbf{d}'}$;
6. if $x \rightsquigarrow_\mathcal{O} y$ and $y \rightsquigarrow_\mathcal{O} z$ then $x \rightsquigarrow_\mathcal{O} z$.

Now, we say that $\mathcal{O}$ is *datatype cyclic* if there is a datatype t such that

$$t_{\mathbf{d}} \rightsquigarrow_{\mathcal{O}} t_{\mathbf{d}'} \rightsquigarrow_{\mathcal{O}} t_{\mathbf{d}}$$

with $\mathbf{d} \neq \mathbf{d}'$. If $\mathcal{O}$ is not datatype cyclic, we say that $\mathcal{O}$ is *datatype acyclic*.

Definition 55 *($\mathcal{O}$-d-acyclic fuzzy p-admissible concrete domain).* Given an assertion free G-$\mathcal{EL}^{+}_{\bot}(\mathbf{D})$ ontology $\mathcal{O}$, a fuzzy p-admissible concrete domain $\mathbf{D}$ is *$\mathcal{O}$-d-acyclic* if $\mathcal{O}$ is datatype acyclic.

Given an assertion free G-$\mathcal{ELO}^{+}_{\bot}(\mathbf{D})$ ontology $\mathcal{O}$, a fuzzy p-admissible concrete domain $\mathbf{D}$ is *$\mathcal{O}$-d-acyclic* if $N(\mathcal{O})$ is datatype acyclic.

Example 56. The fuzzy ontology in Example 50 is datatype cyclic.

It is easily verified that the acyclicity condition in $\mathcal{O}$-d-acyclic concrete domains guarantees that for each $\exists t.\mathbf{d}$ occurring in $\mathcal{O}$ rule $(F\mathbf{D}_2)$ may introduce a new degree β not occurring in $\mathcal{O}$ at most once. Hence, the number of truth degrees that will occur in $\mathsf{fclosure}(\mathcal{O})$ will be bounded by $|\mathcal{O}|$, which allows us to compute $\mathsf{fclosure}(\mathcal{O})$ in polynomial time w.r.t. $|\mathcal{O}|$. The following proposition then extends the results in [49] to finitely-valued and $\mathcal{O}$-d-acyclic fuzzy concrete domains.

Proposition 57. *Proposition 40 holds for nominal safe G-$\mathcal{ELO}^{+}_{\bot}(\mathbf{D})$ ontologies, which may contain domain, range and reflexive role restriction axioms, if $\mathbf{D}$ is either a loose, strict, finitely-valued or $\mathcal{O}$-d-acyclic fuzzy p-admissible concrete domain (where we consider the size of $\mathbf{D}$ as a fixed parameter).*

Runtime p-admissible concrete domains So far, a major restriction related to fuzzy p-admissible concrete domains is the fact that one may not use triangular or trapezoidal membership functions, as usually done in Fuzzy OWL 2 ontology construction. To partially alleviate this restriction, we are going to introduce now a new type of p-admissibility. For ease of exposition, we will first illustrate it in the crisp case and then extend it to the fuzzy case.

So, consider a crisp $\mathcal{EL}^{+}_{\bot}(\mathbf{D})$ ontology $\mathcal{O}$ and consider $\mathsf{closure}(\mathcal{O})$ computed according to the rules in Figs. 2, 3 and 6.

Definition 58 *(Runtime p-admissible).* A concrete domain is *runtime p-admissible w.r.t. $\mathcal{O}$* if the following conditions hold:

1. satisfiability and entailment decision problems indicated in rules $(\mathbf{D}_1)$ and $(\mathbf{D}_2)$ are decidable in polynomial time; and
2. *$\mathcal{O}$-convexity*: if $\mathsf{closure}(\mathcal{O})$ contains

$$A \sqsubseteq \exists t.\mathbf{d}_i \ (0 \leq i \leq n)$$
$$\exists t.\mathbf{d}'_j \sqsubseteq B_j \ (1 \leq j \leq m)$$

and

$$\bigcup_{0 \leq i \leq n} \{(\exists t.\mathbf{d}_i)(a)\} \models (\exists t.\mathbf{d}'_1 \sqcup \ldots \sqcup \exists t.\mathbf{d}'_m)(a)$$

then

$$\bigcup_{0 \leq i \leq n} \{(\exists t.\mathbf{d}_i)(a)\} \models (\exists t.\mathbf{d}'_j)(a)$$

for some $1 \leq j \leq m$.

Essentially, the difference between p-admissibility and runtime p-admissibility w.r.t. $\mathcal{O}$ consists in the convexity condition: in the former case it has to hold independently from any ontology, while in the latter case convexity is restricted

to hold for expressions occurring in $\mathsf{closure}(\mathcal{O})$ only. In fact, $\mathcal{O}$-convexity ensures that rule ($\mathbf{D}_2$) in Fig. 6 does not omit inferences due to 'disjunctive' entailment relations among datatype restrictions after the inference process is completed.

Remark 59. Note that if the $\mathcal{O}$-convexity condition does not hold we may simulate a disjunction on the right hand side of a GCI, in a similar way as it is shown in [1] (see also Remark 45). In fact, suppose $\mathsf{closure}(\mathcal{O})$ contains

$$A \sqsubseteq \exists t.\mathbf{d}_i \ (0 \leq i \leq n)$$
$$\exists t.\mathbf{d}'_j \sqsubseteq B_j \ (1 \leq j \leq m)$$

and

$$\bigcup_{0 \leq i \leq n} \{(\exists t.\mathbf{d}_i)(a)\} \models (\exists t.\mathbf{d}'_1 \sqcup \ldots \sqcup \exists t.\mathbf{d}'_m)(a)$$
$$\bigcup_{0 \leq i \leq n} \{(\exists t.\mathbf{d}_i)(a)\} \not\models (\exists t.\mathbf{d}'_j)(a), \text{ for some } 1 \leq j \leq m$$

hold. Then

$$\mathcal{O} \models A \sqsubseteq B_1 \sqcup \ldots\ldots \sqcup B_m \ ,$$

but

$$\mathcal{O} \not\models A \sqsubseteq B_i \ (1 \leq j \leq m)$$

and, thus, the inference rules for $\mathcal{EL}^+_\perp(\mathbf{D})$ are not complete for this case. To this end, please note that if $\mathcal{O}$ further contains $B_i \sqsubseteq B$ $(1 \leq j \leq m)$, we have $\mathcal{O} \models A \sqsubseteq B$, but $A \sqsubseteq B \notin \mathsf{closure}(\mathcal{O})$.

Anyway, the following follows from the discussion so far:

Proposition 60. *Let $\mathcal{O}$ be an assertion free $\mathcal{EL}^+_\perp(\mathbf{D})$ ontology and let $A, B \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}$. If $\mathbf{D}$ is runtime p-admissible w.r.t. $\mathcal{O}$, then*

1. $\mathsf{closure}(\mathcal{O})$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.
2. $\mathcal{O} \models A \sqsubseteq B$ *iff* $A \sqsubseteq B \in \mathsf{closure}(\mathcal{O})$ *or* $A \sqsubseteq \perp \in \mathsf{closure}(\mathcal{O})$.

Corollary 61. *Proposition 60 holds also for nominal safe $\mathcal{ELO}^+_\perp(\mathbf{D})$ ontologies, which may contain range and reflexive role restriction axioms.*

Example 62. Consider the concrete domain $\mathbf{D}$ with predicates $\{\leq_v, >_v\}$. As illustrated in Remark 45, $\mathbf{D}$ is not convex. Additionally, it is easily verified that $\mathbf{D}$ is not $\mathcal{O}$-convex w.r.t. the ontology $\mathcal{O}$ in Remark 45. On the other hand, $\mathbf{D}$ is $\mathcal{O}'$-convex, where $\mathcal{O}'$ is

$$A \sqsubseteq \exists t. \leq_{10}$$
$$\exists t. >_5 \sqsubseteq B$$
$$C \sqsubseteq \exists t. \leq_5 \ .$$

Note that the $\mathcal{O}'$-convexity of $\mathbf{D}$ has the consequence of not affecting the completeness of the inference rules in the sense that Proposition 60 still holds for $\mathcal{O}'$ (but not for $\mathcal{O}$ in Remark 45).

Runtime fuzzy p-admissible concrete domains We next address the notion of runtime p-admissibility w.r.t. a G-$\mathcal{EL}_{\perp}^{+}(\mathbf{D})$ ontology $\mathcal{O}$. Let $\mathsf{fclosure}(\mathcal{O})$ be computed according to the rules in Figs. 4, 5 and 7.

Definition 63 *(Runtime fuzzy p-admissible).* We will say that a fuzzy concrete domain is *runtime fuzzy p-admissible w.r.t.* $\mathcal{O}$ if the following conditions hold:

1. all satisfiability and entailment decision test made in $\mathsf{fclosure}(\mathcal{O})$, as indicated in rules $(F\mathbf{D}_1)$ and $(F\mathbf{D}_2)$, are polynomial in time; and
2. $\mathcal{O}$*-convexity*: if $\mathsf{fclosure}(\mathcal{O})$ contains

$$\langle A \sqsubseteq \exists t.\mathbf{d}_i, \alpha_i\rangle \ (0 \leq i \leq n)$$
$$\langle \exists t.\mathbf{d}'_j \sqsubseteq B_j, \beta_j\rangle \ (1 \leq j \leq m)$$

and

$$\beta = \mathsf{bed}(\cup_{1\leq i\leq n}\{\langle(\exists t.\mathbf{d}_i)(a), \alpha_i\rangle\}, (\exists t.\mathbf{d}'_1 \sqcup \ldots \sqcup \exists t.\mathbf{d}'_m)(a)) > 0$$

then

$$\beta = \mathsf{bed}(\cup_{1\leq i\leq n}\{\langle(\exists t.\mathbf{d}_i)(a), \alpha_i\rangle\}, (\exists t.\mathbf{d}'_j)(a))$$

for some $1 \leq j \leq m$.

In Remark 48 we have shown that convexity can be tested via optimization problems. However, we still need to show that these optimization problems can be solved in polynomial time. To this end, it suffices to show that the generated optimization problems are indeed *linear programming* optimization problems, that are known to be solvable in polynomial time [37]. In particular, for $\mathbf{d} \in \{ls(q_1, q_2), q_1, q_2), tri(q_1, q_2, q_3), trz(q_1, q_2, q_3, q_4)\}$, it suffices to show that constraints of the form $\mathbf{d}(t) \geq \alpha$ and $\mathbf{d}(t) \leq z$ with $t \in [0, 1], \alpha > 0$ and $z \in (0, 1]$ can be represented as linear programming constraints.

- Let us start with the case $\mathbf{d} := ls(q_1, q_2)$. For $\mathbf{d}(t) \geq \alpha$, as $\alpha > 0$, it is not difficult to see that

$$t \leq q_2 - \alpha(q_2 - q_1)$$

  has to hold, which is clearly a linear programming constraint. Similarly, for $\mathbf{d}(t) \leq z$ we have that

$$t \geq q_2 - z(q_2 - q_1)$$

  has to hold, which is a linear programming constraint as well.
- The case $\mathbf{d} := rs(q_1, q_2)$ can be worked out similarly.
- Consider now the case $\mathbf{d} := tri(q_1, q_2, q_3)$. For $\mathbf{d}(t) \geq \alpha$, as $\alpha > 0$, it is not difficult to see that

$$\alpha(q_2 - q_1) + q_1 \leq t \leq q_3 - \alpha(q_3 - q_2)$$

  has to hold, which is a linear programming constraint. Concerning $\mathbf{d}(t) \leq z$, this condition can be represented as

$$\begin{array}{l} q_1 \leq t \leq q_3 \\ t \leq q_1 + z(q_2 - q_1) + q_3 y \\ t \geq q_3 - z(q_3 - q_2) - (q_3 - q_1)(1 - y) \\ y \in \{0, 1\} \ . \end{array}$$

- At last, let us consider the case $\mathbf{d} := trz(q_1, q_2, q_3, q_4)$. It is not difficult to see that this case parallels the one for triangular functions. In fact, $\mathbf{d}(t) \geq \alpha$ can be encoded as

$$\alpha(q_2 - q_1) + q_1 \leq t \leq q_4 - \alpha(q_4 - q_3) \ ,$$

  while $\mathbf{d}(t) \leq z$, this condition can be represented as

$$
\begin{aligned}
& q_1 \leq t \leq q_4 \\
& t \leq q_1 + z(q_2 - q_1) + q_4 y \\
& t \geq q_4 - z(q_4 - q_3) - (q_4 - q_1)(1 - y) \\
& y \in \{0, 1\} \ .
\end{aligned}
$$

Please notice that in case of triangular and trapezoidal fuzzy concrete domain predicates, we have to introduce a new binary variable and so end up in that case with a *Mixed Integer Linear Programming* optimization problem (MILP), which is known to be exponential in the number of binary variables [55]. So, in general, if we consider a typical fuzzification of a datatype property (using an L-function, an R-function, and 3 triangular functions), for each such datatype property, we introduce at most three binary variables. Taking into account that in practice few datatype properties are usually fuzzified and, thus, it is expected that very few triangular fuzzy concrete domain predicates are involved in an $\mathcal{O}$-convexity test, *i.e.*, , at most three for each datatype t, we may consider their impact as negligible from a practical point of view.

Definition 64 *(Various types of runtime fuzzy p-admissible concrete domains).* Given a fuzzy ontology $\mathcal{O}$, a runtime fuzzy p-admissible concrete domain $\mathbf{D}$ is loose (resp. strict, finitely-valued, $\mathcal{O}$-d-acyclic) if it is a loose (resp. strict, finitely-valued, $\mathcal{O}$-d-acyclic) fuzzy p-admissible concrete domain in which the fuzzy p-admissibility condition is replaced with the runtime fuzzy p-admissibility condition.

Proposition 65. *Proposition 40 holds for nominal safe G-$\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ ontologies, which may contain domain, range and reflexive role restriction axioms, if* $\mathbf{D}$ *is either a loose, strict, finitely-valued, or $\mathcal{O}$-d-acyclic runtime fuzzy p-admissible (where we consider the size of* $\mathbf{D}$ *as a fixed parameter).*

3.5. Reasoning with nominals

As we have seen in Remark 8, the inference rules set presented in [1] is incomplete w.r.t. $\mathcal{EL}$ and, thus, so are those presented in [49]. [35] illustrates an alternative that is complete w.r.t. $\mathcal{ELO}$. In the following, we will adapt and extend [35] to cope with (non-nominal safe) fuzzy ontologies as well: specifically, we will address G-$\mathcal{ELO}_{\bot}^{+}(\mathbf{D})$ ontologies with domain, range and reflexive role restrictions.

As pointed out in [35] (see also Remark 8), a GCI such as

$$A \sqsubseteq \exists r.(B \sqcap \{a\}) \ , \tag{1}$$

besides stating that every instance of A is r-connected to the individual a, it also states that a is an instance of B if A has at least one instance, *i.e.*, is non-empty. In order to express such a *conditional* property, the main idea of [35] is to use *conditional* GCIs, which are of the form

$$E : C \sqsubseteq D \ ,$$

where E, C and D are concepts and with semantics $\mathcal{I} \models E : C \sqsubseteq D$ if $E^{\mathcal{I}} \neq \emptyset$ implies $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Therefore, (1) is equivalent to $A \sqsubseteq \exists r.\{a\}$ and $A : \{a\} \sqsubseteq B$. Note that the latter condition also says that if A is non-empty then B is non-empty, and that B has at least individual a as instance. To track implications between non-emptiness of concepts, [35] further introduces a new type of axioms $C \rightsquigarrow D$ (C, D concepts), called *reachability axioms*, with semantics $\mathcal{I} \models C \rightsquigarrow D$ if $C^{\mathcal{I}} \neq \emptyset$ implies $D^{\mathcal{I}} \neq \emptyset$.

Remark 66. Please note that the definite GCI $C \sqsubseteq D$ implies a conditional GCI $A : C \sqsubseteq D$ for every concept name A, and is equivalent to the conditional GCIs $C : C \sqsubseteq D$ and $\top : C \sqsubseteq D$.

We are going now to extend these two notions to the fuzzy case. Specifically, in the following, given a concept C and a fuzzy interpretation $\mathcal{I}$, we write $C^{\mathcal{I}} \neq \emptyset$ if $C^{\mathcal{I}}(x) > 0$ for some $x \in \Delta^{\mathcal{I}}$. Now,

1. a *conditional fuzzy GCI* is of the form

$$\langle E : C \sqsubseteq D, \alpha \rangle \ ,$$

(FOR_0) $\frac{}{\langle E:\ A \sqsubseteq A, 1\rangle}$: $A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

(FOR_1) $\frac{}{\langle E:\ C \sqsubseteq \top, 1\rangle}$: $C \sqsubseteq \top$ in normal form, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

(FOR_2) $\frac{}{E:\ \phi}$: $\phi \in \mathcal{O}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(FOR_{\sqsubseteq}^1)$ $\frac{\langle E:\ C_1 \sqsubseteq C_2, \alpha_1\rangle\ \langle E:\ C_2 \sqsubseteq C_3, \alpha_2\rangle}{\langle E:\ C_1 \sqsubseteq C_3, \alpha_1 \otimes \alpha_2\rangle}$: $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(FOR_{\sqsubseteq}^2)$ $\frac{\langle E:\ A \sqsubseteq B_i, \alpha_i\rangle (1 \leq i \leq n),\ \langle E:\ B_1 \sqcap \ldots \sqcap B_n \sqsubseteq B, \beta\rangle}{\langle E:\ A \sqsubseteq B, \alpha_1 \otimes \ldots \otimes \alpha_n \otimes \beta\rangle}$: $A, B, B_i \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(FOR_{\exists})$ $\frac{\langle E:\ A \sqsubseteq \exists r.B, \alpha_1\rangle\ \langle E:\ B \sqsubseteq C, \alpha_2\rangle}{\langle E:\ A \sqsubseteq \exists r.C, \alpha_1 \otimes \alpha_2\rangle}$: $A, B, C \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $C \neq \bot$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(FOR_{\bot})$ $\frac{\langle E:\ A \sqsubseteq \exists r.B, \alpha_1\rangle\ \langle E:\ B \sqsubseteq \bot, \alpha_2\rangle}{\langle E:\ A \sqsubseteq \bot, \alpha_1 \otimes \alpha_2\rangle}$: $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

(FOR_H) $\frac{\langle E:\ A \sqsubseteq \exists r.B, \alpha_1\rangle\ \langle r \sqsubseteq s, \alpha_2\rangle}{\langle E:\ A \sqsubseteq \exists s.B, \alpha_1 \otimes \alpha_2\rangle}$: $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(FOR_{H^\star})$ $\frac{\langle r_1 \sqsubseteq r_2, \alpha_1\rangle\ \langle r_2 \sqsubseteq r_3, \alpha_2\rangle}{\langle r_1 \sqsubseteq r_3, \alpha_1 \otimes \alpha_2\rangle}$:

$(FOR_{\circ})$ $\frac{\langle E:\ A_1 \sqsubseteq \exists r_1.B, \alpha_1\rangle\ \langle E:\ B \sqsubseteq \exists r_2.A_2, \alpha_2\rangle\ \langle r_1 \circ r_2 \sqsubseteq s, \alpha_3\rangle}{\langle E:\ A_1 \sqsubseteq \exists s.A_2, \alpha_1 \otimes \alpha_2 \otimes \alpha_3\rangle}$: $A_1, B, A_2 \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(\rightsquigarrow_0)$ $\frac{}{E \rightsquigarrow E}$: $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(\rightsquigarrow_1)$ $\frac{}{E \rightsquigarrow \{a\}}$: $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$, a occurs in $\mathcal{O}$

$(\rightsquigarrow_2)$ $\frac{E \rightsquigarrow A, \langle E:\ A \sqsubseteq \exists r.B, \alpha\rangle}{E \rightsquigarrow B}$: $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

$(\rightsquigarrow_3)$ $\frac{E \rightsquigarrow A, E \rightsquigarrow B, \langle E:\ A \sqsubseteq \{a\}, \alpha_1\rangle, \langle E:\ B \sqsubseteq \{a\}, \alpha_2\rangle}{\langle E:\ A \sqsubseteq B, 1\rangle}$: $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}} \cup \{\bot, \top\}$, $E \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$

Fig. 8. Inference rules for G-$\mathcal{ELO}_{\bot}^{+}$ ontologies in normal form.

where E, C and D are concepts, $\alpha \in (0, 1]$ and with semantics $\mathcal{I} \models \langle E: C \sqsubseteq D, \alpha\rangle$ if $E^{\mathcal{I}} \neq \emptyset$ implies $(C \sqsubseteq D)^{\mathcal{I}} \geq \alpha$;

2. a *reachability axiom* is of the form (C, D concepts)

$$C \rightsquigarrow D$$

with semantics $\mathcal{I} \models C \rightsquigarrow D$ if $C^{\mathcal{I}} \neq \emptyset$ implies $D^{\mathcal{I}} \neq \emptyset$.

We are ready now to present the new inference rules. So, to start with, without loss of generality, *we will assume that all G-$\mathcal{ELO}_{\bot}^{+}$ GCIs are in normal form*. The inference rules for assertion free G-$\mathcal{ELO}_{\bot}^{+}$ are presented in Fig. 8 and are an extension of those presented for $\mathcal{EL}_{\bot}^{+}$ in Fig. 5.

The first ten rules $(FOR_0) - (FOR_{\circ})$ are the natural extension of those in Fig. 5. The remaining rules $(\rightsquigarrow_0) - (\rightsquigarrow_3)$ are an adaption of some rules in [35] to our setting. Specifically,

- Rules $(\rightsquigarrow_0)$ and $(\rightsquigarrow_1)$ are a kind of initialization rules and stem from the initialization conditions established in [35, Theorem 4];
- Rule $(\rightsquigarrow_2)$ is used to infer new reachability axioms via concept inclusion axioms involving existential concept restrictions. Reachability axioms can either be used to infer new reachability axioms, as with rule $(\rightsquigarrow_2)$, or to derive new inclusion axioms via rule $(\rightsquigarrow_3)$. The crisp version of rule $(\rightsquigarrow_2)$ is rule $(R_{\rightsquigarrow}^{+})$ in [35];
- Rule $(\rightsquigarrow_3)$ is the fuzzy variant of rule $(R_{\{\}})$ in [35] and essentially allows to infer new subsumption relations in case concepts are inferred to be nominals (*i.e.*, concepts A and B). Note that the rule is symmetric w.r.t. A and B and, thus, if $\langle E:\ A \sqsubseteq B, 1\rangle$ is inferred then also $\langle E:\ B \sqsubseteq A, 1\rangle$ is inferred as well. The degree of these conditional fuzzy GCIs is 1 as A and B are either empty or a nominal, in any model of the involved ontology.

Now, given a G-$\mathcal{ELO}_{\bot}^{+}$ ontology $\mathcal{O}$, let $\mathsf{fclosure}(\mathcal{O})$ be a set that is closed under the rules in Figs. 8 applied to $\mathcal{O}$. Then the following extension to Proposition 30 can be shown (proof in Appendix A).

Proposition 67. *In G-$\mathcal{ELO}_{\bot}^{+}$, let $\mathcal{O}$ be an assertion free ontology and let $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. Then,*

1. $\mathsf{fclosure}(\mathcal{O})$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.
2. $\mathcal{O} \models \langle A \sqsubseteq B, \alpha \rangle$ *iff one of the following conditions holds:*
   (a) $\langle A\colon A \sqsubseteq B, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$ *with* $\beta \geq \alpha$*;*
   (b) $\langle A\colon A \sqsubseteq \bot, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$*;*
   (c) $\langle A\colon \{a\} \sqsubseteq \bot, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$.

Note that apparently, the crisp analogue of case 2c) in Proposition 67 has not been contemplated in [35, Theorem 4], which may possibly hint to an incompleteness for this very specific case. Anyway, the correction is negligible.[11]

The following example illustrates the behavior from an inference point of view.

Example 68. Let us consider the example in Remark 8, and a normal form of it, *i.e.*, the ontology $\mathcal{O}$ with axioms

(1) $A \sqsubseteq \exists r.C$

(2) $C \sqsubseteq B$

(3) $C \sqsubseteq \{a\}$

(4) $A \sqsubseteq \exists s.\{a\}$

(5) $\exists s.B \sqsubseteq B$.

We know that $\mathcal{O} \models A \sqsubseteq B$ holds. According to Proposition 67, let us show that indeed we get $A\colon A \sqsubseteq B \in \mathsf{fclosure}(\mathcal{O})$. The following inference sequence illustrates it:

(5)	$A \rightsquigarrow A$	$(\rightsquigarrow_0)$ applied to A
(6)	$A\colon A \sqsubseteq \exists r.C$	(FOR_2) applied to (1) and A
(7)	$A \rightsquigarrow C$	$(\rightsquigarrow_2)$ applied to (5), (6)
(8)	$A \rightsquigarrow \{a\}$	$(\rightsquigarrow_1)$ applied to A and $\{a\}$
(9)	$A\colon C \sqsubseteq \{a\}$	(FOR_2) applied to (3) and A
(10)	$A\colon \{a\} \sqsubseteq \{a\}$	(FOR_0) applied to A and $\{a\}$
(11)	$A\colon \{a\} \sqsubseteq C$	$(\rightsquigarrow_3)$ applied to (7) − (10)
(12)	$A\colon C \sqsubseteq B$	(FOR_2) applied to (2) and A
(13)	$A\colon \{a\} \sqsubseteq B$	$(FOR_{\sqsubseteq})$ applied to (11), (12)
(14)	$A\colon A \sqsubseteq \exists s.\{a\}$	(FOR_2) applied to (4) and A
(15)	$A\colon \exists s.B \sqsubseteq B$	(FOR_2) applied to (5) and A
(16)	$A\colon A \sqsubseteq B$	$(FOR_{\exists}^{2})$ applied to (13) − (15)

which concludes.

Dealing with domain, range and reflexive role restrictions in G-$\mathcal{ELO}_{\bot}^{+}$ It is easily verified that the procedure described in Section 3.3 still applies for G-$\mathcal{ELO}_{\bot}^{+}$ ontologies as well as the management of domain, range and reflexive role restrictions is essentially a pre-processing step. Therefore, we have immediately that Proposition 40 extents to G-$\mathcal{ELO}_{\bot}^{+}$ as well.

Corollary 69. *In G-$\mathcal{ELO}_{\bot}^{+}$, let $\mathcal{O}$ be an assertion free fuzzy ontology, which may contain domain, range and reflexive role restriction axioms, and let $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. Then,*

1. $\mathit{bed}(\mathcal{O}, A \sqsubseteq B) = \alpha$ *iff* $\alpha = \max(\beta_1, \beta_2, \beta_3)$ *with*

[11] Note that cases 2b) and 2c) roughly say that "A has no instance" and, thus, the subsumption relation between A and B holds.

$$(F\mathbf{D}_1)\ \frac{\langle E\colon A \sqsubseteq \exists t.\mathbf{d}_1, \alpha_1\rangle, \ldots, \langle E\colon A \sqsubseteq \exists t.\mathbf{d}_n, \alpha_n\rangle}{E\colon A \sqsubseteq \bot}\ :\ \begin{array}{l} A \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}} \cup \{\bot, \top\}, E \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}, \\ a \text{ new individual}, \textit{and} \\ \{\langle(\exists t.\mathbf{d}_1)(a), \alpha_1\rangle, \ldots, \\ \quad \langle(\exists t.\mathbf{d}_n)(a), \alpha_n\rangle\} \text{ unsatisfiable} \end{array}$$

$$(F\mathbf{D}_2)\ \frac{\langle E\colon A \sqsubseteq \exists t.\mathbf{d}_1, \alpha_1\rangle, \ldots, \langle E\colon A \sqsubseteq \exists t.\mathbf{d}_n, \alpha_n\rangle}{\langle E\colon A \sqsubseteq \exists t.\mathbf{d}, \beta\rangle}\ :\ \begin{array}{l} A \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}} \cup \{\bot, \top\}, E \in \mathsf{N}^{\mathcal{O}}_{\mathscr{C}}, \\ \exists t.\mathbf{d} \text{ occurs in } \mathcal{O}, \\ a \text{ new individual}, \textit{and} \\ 0 < \beta = \mathsf{bed}(\cup_{1 \le i \le n}\{\langle(\exists t.\mathbf{d}_i)(a), \alpha_i\rangle\}, \exists t.\mathbf{d}) \end{array}$$

Fig. 9. Inference rules for fuzzy concrete domains in G-$\mathcal{ELO}^+_\bot(\mathbf{D})$.

(a) *if* $\langle A\colon A \sqsubseteq B, \gamma_1\rangle \in \mathsf{fclosure}(\mathcal{O})$ *then* $\beta_1 = \gamma_1$ *else* $\beta_1 = 0$*;*
(b) *if* $\langle A\colon A \sqsubseteq \bot, \gamma_2\rangle \in \mathsf{fclosure}(\mathcal{O})$*, then* $\beta_2 = 1$ *else* $\beta_2 = 0$.
(c) *if* $\langle A\colon \{a\} \sqsubseteq \bot, \gamma_3\rangle \in \mathsf{fclosure}(\mathcal{O})$*, then* $\beta_3 = 1$ *else* $\beta_3 = 0$.

2. *bed*$(\mathcal{O}, A \sqsubseteq B)$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.

Dealing with fuzzy concrete domains in G-$\mathcal{ELO}^+_\bot(\mathbf{D})$ The extension of our calculus involving conditional fuzzy GCIs to G-$\mathcal{ELO}^+_\bot(\mathbf{D})$, *i.e.*, G-$\mathcal{ELO}^+_\bot$ with fuzzy concrete domains, is immediate. In fact, it suffices to replace the two rules in Fig. 7 with the ones in Fig. 9.

The following proposition shows that Propositions 57 and 65 extend to G-$\mathcal{ELO}^+_\bot(\mathbf{D})$ as well.

Proposition 70. *Corollary 69 holds for assertion free G-$\mathcal{ELO}^+_\bot(\mathbf{D})$ ontologies, which may contain domain, range and reflexive role restriction axioms, if* $\mathbf{D}$ *is either a loose, strict, finitely-valued, or $\mathcal{O}$-d-acyclic (runtime) fuzzy p-admissible (where we consider the size of* $\mathbf{D}$ *as a fixed parameter).*

4. Related work

In this section we overview previous works considering fuzzy extensions of logics of the $\mathcal{EL}$ family.

In [56] the authors consider G-$\mathcal{EL}$ with graded concept inclusion axioms and non-graded (crisp) role composition axioms, and without assertion axioms. Inference rules are provided to decide in polynomial time graded subsumption. The algorithm is essentially a fuzzy variant of the subsumption decision algorithm for $\mathcal{EL}^{++}$ presented in [1], which is known to be incomplete, but restricted to $\mathcal{EL}^+$.

In [10] interval-valued fuzzy $\mathcal{EL}^{++}$ is addressed, but with non-graded (crisp) role composition axioms. Here an interval $[\alpha_1, \alpha_2]$ is attached to a crisp $\mathcal{EL}^{++}$ axiom τ to state that the degree of truth of τ is in $[\alpha_1, \alpha_2]$. In principle, this generalizes the usual fuzzy DL case, which can be seen as an interval-valued one with intervals restricted to be of the form $[\alpha, 1]$. In [10], the authors state to rely on Łukasiewicz logic for the semantics of DL operators and also proposes a subsumption algorithm in the style of [56] extended to $\mathcal{EL}^{++}$. However, the results in [10] apparently exhibit various issues. For instance, besides missing various fuzzy variants of rules described in [1], (i) the rules cannot be complete as [1] is known being not complete for unrestricted usage of nominals [39–41] (see also Remark 8); (ii) the rules cannot be complete as the algorithm in [10] runs in polynomial time, contrary to the results in e.g. [17] that state that in Ł-$\mathcal{EL}$ the graded subsumption problem cannot be a polynomially solvable problem (see later on).

In [82], the authors consider the same language as in [56], *i.e.*, G-$\mathcal{EL}$ with graded concept inclusion axioms and non-graded (crisp) role composition axioms, but they provide a MapReduce algorithm to decide graded subsumption.

The series of papers [72,73,76,77] is essentially about fuzzy/ordered $\mathcal{EL}$ with crisp roles and the addition of *aggregation operators* over concepts (see, e.g. [14]) in place of concept conjunction. Aggregation operators are more general than concept conjunction and are used here to model user preferences. However, these works apparently exhibit an intrinsic issue similar to [10]. Indeed, the provided polynomial complexity results seem incorrect: for instance, by choosing an aggregation operator @ to be Łukasiewicz conjunction, the subsumption cannot be decided in polynomial time [17]. Similar problems occur for other choices of @ such as, e.g., disjunction or weighted sum.

In [75], the authors present a paraconsistent variant of a fuzzy $\mathcal{EL}^{++}$ without concrete domains. Paraconsistency is obtained using *bilattices* (see, e.g. [31]). Essentially, a fuzzy interpretation is made of two ingredients: (i) a positive fuzzy interpretation function defining the degree of membership; and (ii) a negative fuzzy interpretation function defining the degree of non-membership. No reasoning algorithm is provided. The approach is essentially an adaption of [58] to the DL case.

Notably, the work [49] is an extension of [56] and addresses G-$\mathcal{EL}^{++}$. Specifically, it extends [56] with nominals and specific fuzzy concrete domains to guarantee polynomial decision algorithms. However, like for [10], the inference algorithm in [49] follows that in [1] and, thus, is incomplete w.r.t. nominals (see also Remark 8). Another question mark concerns inference rules **CR8a** and **CR8b**, which do not clarify how one has to find out the value d referred in them.

The series of works [17,20,21] consider fuzzy $\mathcal{EL}$ in the general case, *i.e.*, in the case any t-norm $\otimes$ is used. These works illustrate, for [0, 1]-valued fuzzy $\mathcal{EL}$, various CO-NP-hard results for: (i) 1-subsumption under Łukasiewicz logic; and (ii) positive subsumption (deciding whether the subsumption degree is always greater than 0) if $\otimes$ starts with Łukasiewicz t-norm. On the other hand, EXPTIME-hardness is shown in case $\otimes$ contains Łukasiewicz t-norm, for: (i) p-subsumption ($p \in (0, 1]$) for [0, 1]-valued fuzzy $\mathcal{EL}$; and (ii) p-subsumption for finitely-valued fuzzy $\mathcal{EL}$. They also show some PTIME-completeness results for: (i) positive-subsumption in [0, 1]-valued fuzzy $\mathcal{EL}$ if $\otimes$ does not start with Łukasiewicz t-norm, (ii) 1-subsumption in [0, 1]-valued fuzzy $\mathcal{EL}$ if $\otimes$ does not start with Łukasiewicz t-norm and if roles are crisp and the inclusion axioms are *normalized*, and (iii) p-subsumption for finitely-valued fuzzy $\mathcal{EL}$ if $\otimes$ does not contain Łukasiewicz t-norm. Moreover, in [18] the focus is on Łukasiewicz t-norm. Among others, it is shown that for Ł-$\mathcal{EL}$, the complexity of reasoning increases from PTIME to EXPTIME, even if only one additional truth value is added to {0, 1}. Also, adding role composition axioms to the language, *i.e.*, in Ł-$\mathcal{EL}^{+}$, the complexity further grows to 2-EXPTIME. On the other hand, if the truth space is [0, 1], reasoning in Ł-$\mathcal{EL}$ becomes surprisingly undecidable. A partial undecidability result regarding the product t-norm is also shown in [18], extending the result in [19]. All these results motivate also why we stick to Gödel logic here to guarantee a polynomial time subsumption decision algorithm.

In [8,9], the authors consider a Datalog rewriting-based approach, inspired by [41], to reason with G-$\mathcal{EL}^{++}$. We envisage here at least two issues: (i) the use of enumerated concepts of the form $\{o_1, \ldots, o_n\}$ (which introduces non-determinism), while neither $\mathcal{EL}^{++}$ nor OWL 2 EL support these constructs (only nominal concept $\{o\}$ is supported); and (ii) the support of generalized fuzzy concrete domains, while the inferences rules do not address this feature. Additionally, as [1,47,49] illustrate, concrete domains have to be somewhat restricted to prevent intractability or even undecidability.

Somewhat less related to our work are [4,29,30] in which $\mathcal{EL}$ is extended with threshold concepts, such as $C_{\geq n}$, where the belonging of an individual a to a concept C is graded by means of a priori fixed interpretation function and then a belongs to the extension of $C_{\geq n}$ if the grade is greater or equal than n. Such expressions, though different, somewhat recall threshold concepts defined in Fuzzy OWL 2 (see [13,15]), but basically their logic is crisp. The authors also provide various computational complexity results related to their logic.

In [42,43,45,44,70] an application using G-$\mathcal{EL}(\mathbf{D})$ is shown, by illustrating how one may learn automatically graded G-$\mathcal{EL}(\mathbf{D})$ GCIs from crisp OWL ontologies. This gives also a partial answer to the typical question related to fuzzy DLs in general: where do the numbers in graded axioms come from.

Eventually, in [11], the author discusses the impact of crisp concepts and roles in some reasoning algorithms for finite fuzzy extensions of OWL 2 EL based on a reduction to classical ontologies.

5. Conclusions and future work

In this work we have addressed the basics of Fuzzy OWL 2 EL under standard and Gödel semantics, which is essentially G-$\mathcal{EL}_{\bot}^{+}$ with nominals, fuzzy concrete domains, domain, range and reflexive role restrictions. We have provided reasoning algorithms and shown that instance/subsumption checking can be decided in polynomial time. We have also shown how to reduce the so-called nominal safe ontologies to ontologies without nominals, which simplifies the inference rules set. We have further addressed fuzzy concrete domains by extending the notion of fuzzy p-admissible concrete domains to some other alternative, useful cases that occur when modeling fuzzy ontologies. Finally, we have identified some issues in related previous work (essentially incompleteness problems).

As future work, we plan to implement, evaluate and optimize our reasoning algorithm within the *fuzzyDL* system [15]. To this end it is also interesting to develop analogue fuzzy inference rules to those, highly optimized, provided in [36] for the crisp case. As already pointed out also for Fuzzy OWL 2 [16], optimizing the inference rules may change radically the running time. Other further directions include the development of the other two Fuzzy OWL 2 profiles, namely Fuzzy OWL 2 QL and Fuzzy OWL 2 RL. Some departing points may be works such as [50–52,54,57,61,63–65,67,71,74]. Here, an interesting point would be to investigate which of the other exten-

sion that belong to Fuzzy OWL 2, such as fuzzy modifiers, aggregation functions, etc., may be migrated, and under which restriction, to the Fuzzy OWL 2 profiles without altering the computation property of their crisp counterpart.

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Appendix A. Proofs

In the following, let us consider the following definition of *fuzzy canonical interpretation*.

Definition 71 *(Fuzzy canonical interpretation).* In G-$\mathcal{EL}_{\bot}^{+}$, let $\mathcal{O}$ be an assertion free fuzzy ontology and assume that there is C such that no $\langle C \sqsubseteq \bot, \beta\rangle$ is in $\mathsf{fclosure}(\mathcal{O})$. Then, the *fuzzy canonical interpretation* $\mathcal{I}$ w.r.t. $\mathcal{O}$ is defined as follows ($\max \emptyset = 0$).

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{x_C \mid \langle C \sqsubseteq \bot, \beta\rangle \notin \mathsf{fclosure}(\mathcal{O}) \text{ for every } \beta > 0\} \\
A^{\mathcal{I}}(x_C) &= \max\{\beta \mid \langle C \sqsubseteq A, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})\} \\
r^{\mathcal{I}}(x_A, x_B) &= \max\{\beta \mid \langle A \sqsubseteq \exists r.B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})\}\ .
\end{aligned}$$

Note that $\mathcal{I}$ is well-defined as $\Delta^{\mathcal{I}} \neq \emptyset$.

Proposition 30. *In G-$\mathcal{EL}_{\bot}^{+}$, let $\mathcal{O}$ be an assertion free fuzzy ontology and let $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. Then,*

1. $\mathsf{fclosure}(\mathcal{O})$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.
2. $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$ *iff there is* $\langle A \sqsubseteq B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ *with* $\beta \geq \alpha$, *or* $\langle A \sqsubseteq \bot, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$.[12]

Proof. *Computational complexity.* As the inference rules in Fig. 4 an Fig. 5 are essentially a fuzzy extension of the inference rules for the crisp case, *i.e.*, Fig. 2 and Fig. 3, from Proposition 11 it follows easily that $\mathsf{fclosure}(\mathcal{O})$ can be computed in polynomial time w.r.t. $|\mathcal{O}|$.

Soundness. Now, it is easily verified that the inference rules in Fig. 4 an Fig. 5 are model preserving in the sense that if an interpretation $\mathcal{I}$ is a model of the antecedent of a rule then $\mathcal{I}$ can be extended to a model of the consequent of that rule.[13] Therefore, the inference rules are sound and, thus, if there is $\langle A \sqsubseteq B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \geq \alpha$, or $\langle A \sqsubseteq \bot, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ for some $\beta > 0$ then $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$.

Completeness. As $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$ iff $\mathcal{O}' \models \langle A \sqsubseteq B, \alpha\rangle$, where $\mathcal{O}'$ is a normal form of $\mathcal{O}$, w.l.o.g. and for ease of presentation we assume that $\mathcal{O}$ is already in normal form.

Now, if $\langle C \sqsubseteq \bot, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta > 0$ for all concepts C then the only if direction holds trivially. So, w.l.o.g. we assume now that there is C such that no $\langle C \sqsubseteq \bot, \beta\rangle$ is in $\mathsf{fclosure}(\mathcal{O})$. Next, consider the fuzzy canonical interpretation $\mathcal{I}$ w.r.t. $\mathcal{O}$ constructed as in Definition 71, which is well-defined as $\Delta^{\mathcal{I}} \neq \emptyset$.

Claim 1. *For every atomic concept A such that no $\langle A \sqsubseteq \bot, \beta\rangle$ is in* $\mathsf{fclosure}(\mathcal{O})$, *$x_A \in \Delta^{\mathcal{I}}$ and $A^{\mathcal{I}}(x_A) = 1$ hold.*

Proof. Consider an atomic concept A such that no $\langle A \sqsubseteq \bot, \beta\rangle$ is in $\mathsf{fclosure}(\mathcal{O})$. Then by construction of $\mathcal{I}$, $x_A \in \Delta^{\mathcal{I}}$. Then by rule (FR_0) we have that $\langle A \sqsubseteq A, 1\rangle \in \mathsf{fclosure}(\mathcal{O})$ and, thus, again by construction of $\mathcal{I}$, $A^{\mathcal{I}}(x_A) = 1$. □

Now, let us prove the following claim.

Claim 2. *For each $x_C \in \Delta^{\mathcal{I}}$ and each concept D, $\langle C \sqsubseteq D, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ for some $\beta > 0$ implies $D^{\mathcal{I}}(x_C) \geq \beta$.*

[12] See Remark 23 for the case $\langle A \sqsubseteq \bot, \beta\rangle$.

[13] The extension of $\mathcal{I}$ is needed for normal form transformations rules only due to the introduction of new atomic concepts.

Proof. The proof is by case analysis on the structure of D. So, assume $x_C \in \Delta^{\mathcal{I}}$ and $\langle C \sqsubseteq D, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ for some $\beta > 0$. Recall that $\langle C \sqsubseteq D, \beta\rangle$ is in normal form.

Case $D = A$**:** As $\langle C \sqsubseteq A, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$, by construction of $A^{\mathcal{I}}$, $A^{\mathcal{I}}(x_C) \geq \beta$.
Case $D = \top$**:** As $\top^{\mathcal{I}}(x) = 1$ for all $x \in \Delta^{\mathcal{I}}$, we have $\top^{\mathcal{I}}(x_C) = 1$.
Case $D = \bot$**:** This case is not possible as by the definition of $\Delta^{\mathcal{I}}$, $\langle C \sqsubseteq \bot, \beta\rangle \notin \mathsf{fclosure}(\mathcal{O})$ for every $\beta > 0$.
Case $D = \exists r.B$**:** Since axioms are in normal form, in this case C is an atomic concept A and $\langle A \sqsubseteq \exists r.B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$. Now, it cannot be the case that $\langle B \sqsubseteq \bot, \alpha\rangle \in \mathsf{fclosure}(\mathcal{O})$, as otherwise by rule $(FR_{\bot})$ we would have $\langle A \sqsubseteq \bot, \beta \otimes \alpha\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \otimes \alpha > 0$, contrary to the assumption that $x_A \in \Delta^{\mathcal{I}}$. So, $x_B \in \Delta^{\mathcal{I}}$. By Claim 1, we have $B^{\mathcal{I}}(x_B) = 1$. Moreover, as $\langle A \sqsubseteq \exists r.B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$, by construction of $r^{\mathcal{I}}$ we also have that $r^{\mathcal{I}}(x_A, x_B) \geq \beta$ and, thus, $(\exists r.B)^{\mathcal{I}}(x_A) \geq r^{\mathcal{I}}(x_A, x_B) \otimes B^{\mathcal{I}}(x_B) = r^{\mathcal{I}}(x_A, x_B) \geq \beta$.

This concludes the proof of the claim. □

Now, we proof the converse of Claim 2.

Claim 3. *Let C and D be two concepts such that $C \sqsubseteq D$ is in normal form. If $D^{\mathcal{I}}(x_C) \geq \alpha > 0$ then there is $\langle C \sqsubseteq D, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \geq \alpha$.*

Proof. The proof is by induction on the structure of D. In each case we assume that $D^{\mathcal{I}}(x_C) \geq \alpha$ and prove that there is $\langle C \sqsubseteq D, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \geq \alpha$.

Case $D = B$**:** As $B^{\mathcal{I}}(x_C) \geq \alpha$, by construction of $\mathcal{I}$ there is $\langle C \sqsubseteq B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \geq \alpha$.
Case $D = \top$**:** By rule (FR_1) we have $\langle C \sqsubseteq \top, 1\rangle \in \mathsf{fclosure}(\mathcal{O})$.
Case $D = \bot$**:** This case cannot occur as $0 = \bot^{\mathcal{I}}(x_A) \geq \alpha > 0$ is not possible.
Case $D = \exists r.B$**:** In this case C is an atomic concept A. As $D^{\mathcal{I}}(x_A) \geq \alpha$, by the definition of the $\exists$ constructor and witnessed property, there is $x_E \in \Delta^{\mathcal{I}}$ such that $\alpha \leq (\exists r.B)^{\mathcal{I}}(x_A) = r^{\mathcal{I}}(x_A, x_E) \otimes B^{\mathcal{I}}(x_E)$ and, thus, $r^{\mathcal{I}}(x_A, x_E) \geq \alpha$ and $B^{\mathcal{I}}(x_E) \geq \alpha$. By definition of $r^{\mathcal{I}}$ there is $\langle A \sqsubseteq \exists r.E, \beta_1\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_1 \geq \alpha$. By induction hypothesis on x_E and B there is $\langle E \sqsubseteq B, \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_2 \geq \alpha$. By the $(FR_{\exists})$ rule $\langle A \sqsubseteq \exists r.B, \beta_1 \otimes \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_1 \otimes \beta_2 \geq \alpha$.

This concludes the proof of the claim. □

Eventually, we proof the following claim.

Claim 4. *$\mathcal{I}$ is a witnessed model of $\mathcal{O}$.*

Proof. We prove that $\mathcal{I}$ satisfies each axiom in $\mathcal{O}$.

Case $\langle D_1 \sqsubseteq D_2, \beta_2\rangle$**.** Consider $x_C \in \Delta^{\mathcal{I}}$ and assume $D_1^{\mathcal{I}}(x_C) = \alpha$. We have to show that $D_2^{\mathcal{I}}(x_C) \geq D_1^{\mathcal{I}}(x_C) \otimes \beta_2 = \alpha \otimes \beta_2$. Now, by Claim 3 there is $\langle C \sqsubseteq D_1, \beta_1\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_1 \geq \alpha$. By the $(FR^{\bot}_{\sqsubseteq})$ rule we have $\langle C \sqsubseteq D_2, \beta_1 \otimes \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$. Eventually, by Claim 2, we have ${D_2}^{\mathcal{I}}(x_C) \geq \beta_1 \otimes \beta_2 \geq \alpha \otimes \beta_2$.
Case $\langle r \sqsubseteq s, \beta_2\rangle$**.** Assume $r^{\mathcal{I}}(x_A, x_B) = \alpha$. We have to show that $s^{\mathcal{I}}(x_A, x_B) \geq \alpha \otimes \beta_2$. By definition of $r^{\mathcal{I}}$, $\langle A \sqsubseteq \exists r.B, \beta_1\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_1 \geq \alpha$. By the (FR_H) rule, $\langle A \sqsubseteq \exists s.B, \beta_1 \otimes \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$ and, thus, by definition of $s^{\mathcal{I}}$, $s^{\mathcal{I}}(x_A, x_B) \geq \beta_1 \otimes \beta_2 \geq \alpha \otimes \beta_2$.
Case $\langle r_1 \circ r_2 \sqsubseteq s, \beta_2\rangle$**.** Assume $(r_1 \circ r_2)^{\mathcal{I}}(x_A, x_B) = \alpha$ holds. We have to show that $s^{\mathcal{I}}(x_A, x_B) \geq \alpha \otimes \beta_2$. As $(r_1 \circ r_2)^{\mathcal{I}}(x_A, x_B) = \alpha$ there is $x_D \in \Delta^{\mathcal{I}}$ such that ${r_1}^{\mathcal{I}}(x_A, x_D) = \alpha_1$ and ${r_2}^{\mathcal{I}}(x_D, x_B) = \alpha_2$ with $\alpha_1 \otimes \alpha_2 = \alpha$. Therefore, by definition of ${r_i}^{\mathcal{I}}$, $\langle A \sqsubseteq \exists r_1.D, \alpha_1\rangle \in \mathsf{fclosure}(\mathcal{O})$ and $\langle D \sqsubseteq \exists r_2.B, \alpha_2\rangle \in \mathsf{fclosure}(\mathcal{O})$. Then, by the $(FR_{\circ})$ rule, $\langle A \sqsubseteq \exists s.B, \alpha \otimes \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$ follows (recall that $\alpha_1 \otimes \alpha_2 = \alpha$). Therefore, by definition of $s^{\mathcal{I}}$, we have that $s^{\mathcal{I}}(x_A, x_B) \geq \alpha \otimes \beta_2$.

Eventually, as $\mathsf{fclosure}(\mathcal{O})$ is finite, the set of truth values involved in the construction of $\mathcal{I}$ is finite and, thus, $\mathcal{I}$ is witnessed. This concludes the proof of the claim. $\square$

We are ready now to complete the whole proof of the proposition. So, assume $\mathcal{O} \models \langle A \sqsubseteq B, \alpha \rangle$. If for some $\beta > 0$, $\langle A \sqsubseteq \bot, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$ then the proposition trivially holds. Otherwise, assume $\langle A \sqsubseteq \bot, \beta \rangle \notin \mathsf{fclosure}(\mathcal{O})$ for all $\beta > 0$. Therefore, the fuzzy canonical interpretation $\mathcal{I}$ is well-founded as $\Delta^{\mathcal{I}} \neq \emptyset$ and by Claim 4, $\mathcal{I} \models \mathcal{O}$. As there is no $\langle A \sqsubseteq \bot, \beta \rangle$ in $\mathsf{fclosure}(\mathcal{O})$, by Claim 1, $x_A \in \Delta^{\mathcal{I}}$ and $A^{\mathcal{I}}(x_A) = 1$. As $\mathcal{I} \models \langle A \sqsubseteq B, \alpha \rangle$, we have that $B^{\mathcal{I}}(x_A) \geq \alpha \otimes A^{\mathcal{I}}(x_A) = \alpha$. Therefore, by Claim 3, there is $\langle A \sqsubseteq B, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \geq \alpha$, which concludes. $\square$

Proposition 31. *In G-$\mathcal{ELO}^+_\bot$, let $\mathcal{O}$ be a fuzzy ontology and ϕ an axiom that do not contain atomic concepts of the form N_a. Then,*

1. *if $N(\mathcal{O}) \models N(\phi)$ then $\mathcal{O} \models \phi$;*
2. *if $N(\mathcal{O}) \models N_a \sqsubseteq \bot$ for some a then $\mathcal{O}$ is inconsistent.*

Proof. The proof is essentially the same as for [36, Lemma 5].

1. Assume $N(\mathcal{O}) \models N(\phi)$ holds, but $\mathcal{O} \not\models \phi$. Then, there is a fuzzy interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{O}$, but $\mathcal{I} \not\models \phi$. Let us define a fuzzy interpretation $\mathcal{J}$ by setting $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$, $N_a^{\mathcal{J}} = \{a^{\mathcal{I}}\}$, $A^{\mathcal{J}} = A^{\mathcal{I}}$ for $A \neq N_a$ (for all a) and $R^{\mathcal{J}} = R^{\mathcal{I}}$. As $N(\cdot)$ merely replaces each $\{a\}$ with N_a and $N_a^{\mathcal{J}} = \{a^{\mathcal{I}}\}$, for every axiom ψ that does not contain atomic concepts of the form N_a we have $\mathcal{I} \models \psi$ iff $\mathcal{J} \models N(\psi)$. As $\mathcal{I} \models \mathcal{O}$ and $\mathcal{I} \not\models \phi$, it follows that $\mathcal{J} \models N(\mathcal{O})$ and $\mathcal{J} \not\models N(\phi)$, which contradicts the assumption $N(\mathcal{O}) \models N(\phi)$.
2. Assume $N(\mathcal{O}) \models N_a \sqsubseteq \bot$ for some a. Then, by point 1. ($\phi := \{a\} \sqsubseteq \bot$), $\mathcal{O} \models \{a\} \sqsubseteq \bot$ follows, *i.e.*, $\mathcal{O}$ is inconsistent. $\square$

Proposition 32. *In G-$\mathcal{ELO}^+_\bot$, let $\mathcal{O}$ be a nominal safe fuzzy ontology and ϕ a safe GCI that do not contain atomic concepts of the form N_a. Assume that $N(\mathcal{O}) \not\models N_a \sqsubseteq \bot$ for all a. Then,*

1. *$\mathcal{O}$ is consistent;*
2. *if $\mathcal{O} \models \phi$ then $N(\mathcal{O}) \models N(\phi)$.*

Proof. The proof is an adaption of [36, Theorem 4]. By applying the rules in Fig. 4 to $N(\mathcal{O})$, w.l.o.g. we may assume that $N(\mathcal{O})$ is already in normal form. Now, assume that $N(\mathcal{O}) \not\models N_a \sqsubseteq \bot$ for all a. Let $\mathcal{J}$ be the fuzzy canonical interpretation w.r.t. $\mathsf{fclosure}(N(\mathcal{O}))$. Note that $\mathcal{J}$ is well-defined and $\Delta^{\mathcal{J}}$ contains, for each individual a, a distinguished individual x_{N_a} with, by Claim 1 applied to $\mathcal{J}$, $N_a^{\mathcal{J}}(x_{N_a}) = 1$. By Claim 4 applied to $\mathcal{J}$, $\mathcal{J} \models N(\mathcal{O})$. Now, let us define a fuzzy interpretation $\mathcal{I}$ with $\Delta^{\mathcal{I}} = \Delta^{\mathcal{J}}$, $A^{\mathcal{I}} = A^{\mathcal{J}}$ for atomic concept A, $r^{\mathcal{I}} = r^{\mathcal{J}}$ for role r, and $a^{\mathcal{I}} = x_{N_a}$ for individual a. Then,

Claim 5. *For every $\mathcal{ELO}^+_\bot$ concept D, for all $x_C \in \Delta^{\mathcal{I}}$:*

1. *if D is safe then $D^{\mathcal{I}}(x_C) = N(D)^{\mathcal{J}}(x_C)$;*
2. *if D is n-safe then $D^{\mathcal{I}}(x_C) \leq N(D)^{\mathcal{J}}(x_C)$.*

Proof. The proof of point 1. is by induction on the structure of D. The only non-trivial case of the induction is for $D = \exists r.\{a\}$. At first, let us show that $(\exists r.\{a\})^{\mathcal{I}}(x_C) \leq (\exists r.N_a)^{\mathcal{J}}(x_C)$. As $a^{\mathcal{I}} = N_a$, $(\exists r.\{a\})^{\mathcal{I}}(x_C) = r^{\mathcal{I}}(x_C, a^{\mathcal{I}}) = r^{\mathcal{I}}(x_C, x_{N_a}) = r^{\mathcal{J}}(x_C, x_{N_a}) = r^{\mathcal{J}}(x_C, x_{N_a}) \otimes N_a^{\mathcal{J}}(x_{N_a}) \leq (\exists r.N_a)^{\mathcal{J}}(x_C)$ holds. Vice-versa, let us show that $(\exists r.\{a\})^{\mathcal{I}}(x_C) \geq (\exists r.N_a)^{\mathcal{J}}(x_C)$. Consider $x_C \in \Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$. If $(\exists r.N_a)^{\mathcal{J}}(x_C) = 0$ then the inequality holds trivially. Otherwise, let us assume $(\exists r.N_a)^{\mathcal{J}}(x_C) > 0$. By definition of $\mathcal{J}$ and its witnessed model property, there is $x_E \in \Delta^{\mathcal{J}}$ with $(\exists r.N_a)^{\mathcal{J}}(x_C) = r^{\mathcal{J}}(x_C, x_E) \otimes N_a^{\mathcal{J}}(x_E)$. By definition of $r^{\mathcal{J}}$ there is $\langle C \sqsubseteq \exists r.E, \beta_1 \rangle \in \mathsf{fclosure}(N(\mathcal{O}))$ with $\beta_1 = r^{\mathcal{J}}(x_C, x_E)$. By definition of $N_a^{\mathcal{J}}$, there is $\langle E \sqsubseteq N_a, \beta_2 \rangle \in \mathsf{fclosure}(N(\mathcal{O}))$ with $\beta_2 = N_a^{\mathcal{J}}(x_E)$. Therefore, by rule $(FR_\exists)$, $\langle C \sqsubseteq \exists r.N_a, \beta_1 \otimes \beta_2 \rangle \in \mathsf{fclosure}(N(\mathcal{O}))$. Therefore, by construction of $\mathcal{J}$, $r^{\mathcal{J}}(x_C, x_{N_a}) \geq \beta_1 \otimes \beta_2 = r^{\mathcal{J}}(x_C, x_E) \otimes N_a^{\mathcal{J}}(x_E)$. As $a^{\mathcal{I}} = x_{N_a}$, $r^{\mathcal{I}} = r^{\mathcal{J}}$, $(\exists r.\{a\})^{\mathcal{I}}(x_C) = r^{\mathcal{I}}(x_C, a^{\mathcal{I}}) = r^{\mathcal{I}}(x_C, x_{N_a}) = r^{\mathcal{J}}(x_C, x_{N_a}) \geq$

$r^{\mathcal{J}}(x_C, x_E) \otimes N_a^{\mathcal{J}}(x_E) = (\exists r.N_a)^{\mathcal{J}}(x_C)$. Therefore, we have $(\exists r.\{a\})^{\mathcal{I}}(x_C) = (\exists r.N_a)^{\mathcal{J}}(x_C)$, which concludes point 1.

Let us now prove point 2. If D is safe, point 2. follows from point 1. Otherwise, let $D = \{a\}$ and $N(D) = N_a$. If $\{a\}^{\mathcal{I}}(x_C) = 0$ then point 2. holds trivially. Otherwise, assume $\{a\}^{\mathcal{I}}(x_C) > 0$, that is, $\{a\}^{\mathcal{I}}(x_C) = 1$ and, thus, $x_C = a^{\mathcal{I}} = x_{N_a}$. In this case, by Claim 1 applied to $\mathcal{J}$, we have $N_a^{\mathcal{J}}(x_C) = N_a^{\mathcal{J}}(x_{N_a}) = 1 = \{a\}^{\mathcal{I}}(x_{N_a}) = \{a\}^{\mathcal{I}}(x_C)$, which concludes. □

We now show that $\mathcal{I} \models \mathcal{O}$. In fact, for role inclusions or composition axioms $\phi \in \mathcal{O}$, since $\mathcal{J} \models N(\mathcal{O})$, $N(\phi) = \phi$ and $\mathcal{I}$ interprets roles as $\mathcal{J}$, we have $\mathcal{I} \models \phi$. It remains to show that $\mathcal{I} \models \phi$ for all fuzzy GCIs $\phi \in \mathcal{O}$. So, let $\phi \in \mathcal{O}$ be of the form $\langle C \sqsubseteq D, \alpha\rangle$. We have to show that for $x \in \Delta^{\mathcal{I}}$, $D^{\mathcal{I}}(x) \geq \alpha \otimes C^{\mathcal{I}}(x)$ holds. By assumption, ϕ is safe, that is C is n-safe and D is safe. Then, for $x \in \Delta^{\mathcal{I}}$ by Claim 5, $C^{\mathcal{I}}(x) \leq N(C)^{\mathcal{J}}(x)$ and $D^{\mathcal{I}}(x) = N(D)^{\mathcal{J}}(x)$. Moreover, as $\mathcal{J} \models N(\mathcal{O})$ and $N(\phi) \in N(\mathcal{O})$, $\mathcal{J} \models N(\phi)$ holds. That is, $N(D)^{\mathcal{J}}(x) \geq N(C)^{\mathcal{J}}(x) \otimes \alpha$. As a consequence, $D^{\mathcal{I}}(x) = N(D)^{\mathcal{J}}(x) \geq N(C)^{\mathcal{J}}(x) \otimes \alpha \geq C^{\mathcal{I}}(x) \otimes \alpha$ and, thus, $\mathcal{I} \models \phi$. As a consequence, $\mathcal{I}$ is a model of $\mathcal{O}$ and, thus, $\mathcal{I}$ is consistent, which concludes the proof of point 1. of the proposition.

Let us now prove point 2. So, let ϕ be a safe fuzzy GCI $\langle C \sqsubseteq D, \alpha\rangle$ with $\mathcal{O} \models \phi$ and $\alpha > 0$. Without loss of generality, we may assume that D is an atomic concept, while C is either a nominal or an atomic concept. In fact, it is easily verified that $\mathcal{O} \models \langle C \sqsubseteq D, \alpha\rangle$ iff $\mathcal{O} \cup \{B \sqsubseteq D\} \models \langle C \sqsubseteq B, \alpha\rangle$, where B is a new atomic concept and, thus, $B \sqsubseteq D$ is safe. If further C is not a nominal, as C is safe, we have $\mathcal{O} \cup \{B \sqsubseteq D\} \models \langle C \sqsubseteq B, \alpha\rangle$ iff $\mathcal{O} \cup \{A \sqsubseteq C, B \sqsubseteq D\} \models \langle A \sqsubseteq B, \alpha\rangle$, where A is a new atomic concept and, thus, $A \sqsubseteq C$ is safe.

So, let $\mathcal{O}$ be safe and let ϕ be a fuzzy GCI $\langle C \sqsubseteq D, \alpha\rangle$, with $\mathcal{O} \models \phi$, where D is an atomic concept B and C is either a nominal $\{a\}$ or an atomic concept A. Now, if $\langle N(C) \sqsubseteq \bot, \alpha\rangle \in \mathsf{fclosure}(N(\mathcal{O}))$ then, by Proposition 30, $N(\mathcal{O}) \models N(\phi)$. Otherwise, assume there is no $\langle N(C) \sqsubseteq \bot, \alpha\rangle \in \mathsf{fclosure}(N(\mathcal{O}))$. Therefore, by construction of $\mathcal{J}$ and $\mathcal{I}$, $x_{N(C)} \in \Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$. Moreover, as $\mathcal{O} \models \phi$ and $\mathcal{I} \models \mathcal{O}$, $\mathcal{I} \models \phi$ follows. Now, we analyze the two cases of the form of C.

Case $C = A$. If C is an atom A then $A = N(A), x_A = x_{N(A)}$ and, thus, by Claim 1 applied to $\mathcal{J}$, $A^{\mathcal{I}}(x_A) = A^{\mathcal{J}}(x_A) = 1$. As $\mathcal{I} \models \phi$, $B^{\mathcal{J}}(x_A) = B^{\mathcal{I}}(x_A) \geq \alpha \otimes A^{\mathcal{I}}(x_A) = \alpha > 0$ follows. Therefore, by Claim 3 applied to $\mathcal{J}$, there is $\langle A \sqsubseteq B, \beta\rangle \in \mathsf{fclosure}(N(\mathcal{O}))$ with $\beta \geq \alpha$.

Case $C = \{a\}$. Assume C is a nominal $\{a\}$. As $\mathcal{I} \models \phi$ and $a^{\mathcal{I}} = x_{N_a}$, $B^{\mathcal{J}}(x_{N_a}) = B^{\mathcal{I}}(a^{\mathcal{I}}) \geq \alpha > 0$ follows. Again, by Claim 3 applied to $\mathcal{J}$, there is $\langle N_a \sqsubseteq B, \beta\rangle \in \mathsf{fclosure}(N(\mathcal{O}))$ with $\beta \geq \alpha$.

Therefore, in both cases above, by Proposition 30, $N(\mathcal{O}) \models N(\phi)$ follows, which concludes the proof. □

Next we address Proposition 37. We start with the following Lemma.

Lemma 72. *Consider Gödel logic and suppose $\mathcal{I} \models \mathsf{ran}(r) \sqsubseteq A$, then for all $y \in \Delta^{\mathcal{I}}$, $A^{\mathcal{I}}(y) \geq \sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y)$.*

Proof. Assume $\mathcal{I} \models \mathsf{ran}(r) \sqsubseteq A$. Then by definition,

$$\begin{aligned}
&\inf_{x \in \Delta^{\mathcal{I}}} \{1 \Rightarrow \inf_{y \in \Delta^{\mathcal{I}}} \{r^{\mathcal{I}}(x, y) \Rightarrow A^{\mathcal{I}}(y)\}\} = 1 \\
\text{iff } &\inf_{x \in \Delta^{\mathcal{I}}} \inf_{y \in \Delta^{\mathcal{I}}} \{r^{\mathcal{I}}(x, y) \Rightarrow A^{\mathcal{I}}(y)\} = 1 \\
\text{iff } &\inf_{y \in \Delta^{\mathcal{I}}} \inf_{x \in \Delta^{\mathcal{I}}} \{r^{\mathcal{I}}(x, y) \Rightarrow A^{\mathcal{I}}(y)\} = 1 \ .
\end{aligned}$$

Therefore, for all $y \in \Delta^{\mathcal{I}}$,

$$\inf_{x \in \Delta^{\mathcal{I}}} \{r^{\mathcal{I}}(x, y) \Rightarrow A^{\mathcal{I}}(y)\} = 1 \ . \tag{A.1}$$

Next, we show that indeed Eq. (A.1) implies that

$$\sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \leq A^{\mathcal{I}}(y) \tag{A.2}$$

holds for all $y \in \Delta^{\mathcal{I}}$.

So, assume Eq. (A.1) holds. Therefore, for all $x, y \in \Delta^{\mathcal{I}}$, $r^{\mathcal{I}}(x, y) \Rightarrow A^{\mathcal{I}}(y) = 1$. Now, for $y \in \Delta^{\mathcal{I}}$, suppose to the contrary that $\sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) > A^{\mathcal{I}}(y)$ holds. Then there is $x \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) > A^{\mathcal{I}}(y)$ and, thus, $1 = r^{\mathcal{I}}(x, y)) \Rightarrow A^{\mathcal{I}}(y) = A^{\mathcal{I}}(y) < r^{\mathcal{I}}(x, y)$, which cannot be the case. Therefore, we have $\sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \leq A^{\mathcal{I}}(y)$ and, thus, Eq. (A.2) holds, which concludes the proof. □

Now, we are ready to proof Proposition 37.

Proposition 37. *In G-$\mathcal{EL}_{\bot}^{+}$, let $\mathcal{O}$ be an assertion free fuzzy ontology, which may contain range and reflexive role restriction axioms, and $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. Then, $\mathcal{O} \models \langle A \sqsubseteq B, \alpha \rangle$ iff $\mathcal{O}' \models \langle A \sqsubseteq B, \alpha \rangle$, where $\mathcal{O}'$ has been determined by the* **RangeElimination** *step.*

Proof. For the $\Leftarrow$ direction assume $\mathcal{O}' \models \langle A \sqsubseteq B, \alpha \rangle$ and consider $\mathcal{I} \models \mathcal{O}$. Let us extend $\mathcal{I}$ to interpretation $\mathcal{J}$, by setting $\Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$ and interpreting

$$X_{r,B}^{\mathcal{J}}(y) = \sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes B^{\mathcal{I}}(y) \ .$$

Let us show that $\mathcal{J} \models \mathcal{O}'$. It suffices to show that $\mathcal{J}$ satisfies the new axioms added during the **RangeElimination** step. So, consider $\langle C \sqsubseteq \exists r.B, \alpha \rangle \in \mathcal{O}$.

Case $\langle C \sqsubseteq \exists r.X_{r,B}, \alpha \rangle$**.** We have $\mathcal{I} \models \langle C \sqsubseteq \exists r.B, \alpha \rangle$. Now, for all $x \in \Delta^{\mathcal{J}} = \Delta^{\mathcal{I}}$

$$\begin{aligned}
(\exists r.X_{r,B})^{\mathcal{J}}(x) &= \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes X_{r,B}^{\mathcal{J}}(y) \\
&= \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes \sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(z, y) \otimes B^{\mathcal{I}}(y) \\
&= \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes B^{\mathcal{I}}(y) \\
&= (\exists r.B)^{\mathcal{I}}(x) \ ,
\end{aligned}$$

and, thus, $\mathcal{J} \models \langle C \sqsubseteq \exists r.X_{r,B}, \alpha \rangle$, which concludes.

Case $\langle X_{r,B} \sqsubseteq B, 1 \rangle$**.** We have $X_{r,B}^{\mathcal{J}}(y) = \sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes B^{\mathcal{I}}(y) \leq B^{\mathcal{I}}(y)$ and, thus, $\mathcal{J} \models \langle X_{r,B} \sqsubseteq B, 1 \rangle$.

Case $\langle X_{r,B} \sqsubseteq A, 1 \rangle$**.** Consider $A \in \mathsf{ran}_{\mathcal{O}}(r)$. It suffices to show that for all $y \in \Delta^{\mathcal{I}}$, $A^{\mathcal{I}}(y) \geq X_{r,B}^{\mathcal{J}}(y) = \sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes B^{\mathcal{I}}(y)$ holds. As $\mathcal{I} \models \mathcal{O}$, we have $\mathcal{I} \models \mathsf{ran}(r) \sqsubseteq A$ and, thus, by Lemma 72 we have

$$\begin{aligned}
A^{\mathcal{J}}(y) &= A^{\mathcal{I}}(y) \\
&\geq \sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \\
&\geq \sup_{x \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes B^{\mathcal{I}}(y) \\
&= X_{r,B}^{\mathcal{J}}(y) \ ,
\end{aligned}$$

which concludes.

Now, from $\mathcal{J} \models \mathcal{O}'$, by assumption $\mathcal{J} \models \langle A \sqsubseteq B, \alpha \rangle$ follows. As $\mathcal{J}$ is the same as $\mathcal{I}$ on A and B, $\mathcal{I} \models \langle A \sqsubseteq B, \alpha \rangle$ holds, and, thus, $\mathcal{O} \models \langle A \sqsubseteq B, \alpha \rangle$, which concludes the $\Leftarrow$ direction.

For the $\Rightarrow$ direction, assume $\mathcal{O}' \not\models \langle A \sqsubseteq B, \alpha \rangle$. We will show that $\mathcal{O} \not\models \langle A \sqsubseteq B, \alpha \rangle$. From $\mathcal{O}' \not\models \langle A \sqsubseteq B, \alpha \rangle$ there is $\mathcal{J} \models \mathcal{O}'$ s.t. $\mathcal{J} \not\models \langle A \sqsubseteq B, \alpha \rangle$. Note, $\mathcal{J}$ may not be a model of $\mathcal{O}$ as it may not satisfy some range restriction in $\mathcal{O}$. We are going to define now an interpretation $\mathcal{I}$, built from $\mathcal{J}$, that will be a model of $\mathcal{O}$. Specifically, let $\mathcal{I}$ be defined as $\mathcal{J}$ except that $\mathcal{I}$ interprets role differently: for $x, y \in \Delta^{I} = \Delta^{\mathcal{J}}$ let

$$r^{\mathcal{I}}(x, y) = r^{\mathcal{J}}(x, y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(y) \ .$$

To show that $\mathcal{I} \models \mathcal{O}$, we only consider the axioms in $\mathcal{O}$ that may be influenced by the definition of $\mathcal{I}$.

Case $\langle C \sqsubseteq \exists r.B, \alpha\rangle \in \mathcal{O}$**.** In this case $\mathcal{O}'$ contains $\langle C \sqsubseteq \exists r.X_{r,B}, \alpha\rangle$, $\langle X_{r,B} \sqsubseteq B, 1\rangle$, and $\langle X_{r,B} \sqsubseteq A, 1\rangle$ for all $A \in \mathsf{ran}_{\mathcal{O}}(r)$. Consider $x \in \Delta^{\mathcal{I}}$. Then by construction of $\mathcal{I}$ and by the fact that $\mathcal{J}$ satisfies the former three type of axioms, we have

$$\begin{aligned}
\alpha \otimes C^{\mathcal{I}}(x) &= \alpha \otimes C^{\mathcal{J}}(x) \\
&\leq (\exists r.X_{r,B})^{\mathcal{J}}(x) \\
&= \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{J}}(x,y) \otimes X_{r,B}^{\mathcal{J}}(y) \\
&\leq \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{J}}(x,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(y) \otimes B^{\mathcal{J}}(y) \\
&= \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \otimes B^{\mathcal{I}}(y) \\
&= (\exists r.B)^{\mathcal{I}}(x) \ ,
\end{aligned}$$

which concludes this case.

Case $\langle \exists r.A \sqsubseteq B, \alpha\rangle \in \mathcal{O}$**.** In this case, $\langle \exists r.A \sqsubseteq B, \alpha\rangle \in \mathcal{O}'$ and, thus, $\mathcal{J} \models \langle \exists r.A \sqsubseteq B, \alpha\rangle$. Now, for $x \in \Delta^{\mathcal{I}}$ we have

$$\begin{aligned}
B^{\mathcal{I}}(x) &= B^{\mathcal{J}}(x) \\
&\geq \alpha \otimes (\exists r.A)^{\mathcal{J}}(x) \\
&= \alpha \otimes \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{J}}(x,y) \otimes A^{\mathcal{J}}(y) \\
&\geq \alpha \otimes \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{J}}(x,y) \otimes A^{\mathcal{J}}(y) \otimes (\bigsqcap_{E \in \mathsf{ran}_{\mathcal{O}}(r)} E)^{\mathcal{J}}(y) \\
&= \alpha \otimes \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \otimes A^{\mathcal{I}}(y) \\
&= \alpha \otimes (\exists r.A)^{\mathcal{I}}(x) \ ,
\end{aligned}$$

which concludes this case.

Case $\epsilon \sqsubseteq r \in \mathcal{O}$**.** In this case $\epsilon \sqsubseteq r \in \mathcal{O}'$, $\top \sqsubseteq A \in \mathcal{O}'$ for all $A \in \mathsf{ran}_{\mathcal{O}}(r)$, and, thus, $\mathcal{J}$ satisfies all these axioms. Therefore, $\forall x \in \Delta^{\mathcal{J}}$ we have that $r^{\mathcal{J}}(x,x) = 1$ and, as $\top \sqsubseteq A \in \mathcal{O}'$, $1 = A^{\mathcal{J}}(x) = A^{\mathcal{I}}(x)$ holds. Therefore,

$$\begin{aligned}
r^{\mathcal{I}}(x,x) &= r^{\mathcal{J}}(x,x) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(x) \\
&= r^{\mathcal{J}}(x,x) \\
&= 1 \ ,
\end{aligned}$$

which concludes this case.

Case $\langle s \sqsubseteq r, \alpha\rangle \in \mathcal{O}$**.** In this case $\langle s \sqsubseteq r, \alpha\rangle \in \mathcal{O}'$ and, thus, $\mathcal{J} \models \langle s \sqsubseteq r, \alpha\rangle$ and, by definition, $\mathsf{ran}_{\mathcal{O}}(r) \subseteq \mathsf{ran}_{\mathcal{O}}(s)$. Then, by definition of $\mathcal{I}$, for $x, y \in \Delta^{\mathcal{I}}$ we have

$$\begin{aligned}
\alpha \otimes s^{\mathcal{I}}(x,y) &= \alpha \otimes s^{\mathcal{J}}(x,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(s)} A)^{\mathcal{J}}(y) \\
&\leq \alpha \otimes s^{\mathcal{J}}(x,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(y) \\
&\leq r^{\mathcal{J}}(x,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(y) \\
&= r^{\mathcal{I}}(x,y) \ ,
\end{aligned}$$

which concludes this case.

Case $\langle r_1 \circ r_2 \sqsubseteq r, \alpha\rangle \in \mathcal{O}$**.** In this case $\langle r_1 \circ r_2 \sqsubseteq r, \alpha\rangle \in \mathcal{O}'$ and, thus, $\mathcal{J} \models \langle r_1 \circ r_2 \sqsubseteq r, \alpha\rangle$ and from the syntactic constraint $(\star_f)$, we have that $\mathsf{ran}_{\mathcal{O}}(r) \subseteq \mathsf{ran}_{\mathcal{O}}(r_2)$. Now, by definition of $\mathcal{I}$ we have

$$\begin{aligned}
\alpha \otimes (r_1 \circ r_2)^{\mathcal{I}}(x,y) &= \alpha \otimes \sup_{z \in \Delta^{\mathcal{I}}} r_1^{\mathcal{I}}(x,z) \otimes r_2^{\mathcal{I}}(z,y) \\
&= \alpha \otimes \sup_{z \in \Delta^{\mathcal{I}}} r_1^{\mathcal{J}}(x,z) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r_1)} A)^{\mathcal{J}}(z) \\
&\qquad \otimes r_2^{\mathcal{J}}(z,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r_2)} A)^{\mathcal{J}}(y) \\
&\leq \alpha \otimes \sup_{z \in \Delta^{\mathcal{I}}} r_1^{\mathcal{J}}(x,z) \otimes r_2^{\mathcal{J}}(z,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(y) \\
&\leq \alpha \otimes (r_1 \circ r_2)^{\mathcal{J}}(x,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(y) \\
&\leq r^{\mathcal{J}}(x,y) \otimes (\bigsqcap_{A \in \mathsf{ran}_{\mathcal{O}}(r)} A)^{\mathcal{J}}(y) \\
&= r^{\mathcal{I}}(x,y) \ ,
\end{aligned}$$

which concludes this case.

Case $\mathsf{ran}(r) \sqsubseteq A \in \mathcal{O}$**.** It suffices to show that for all $x, y \in \Delta^{\mathcal{I}}$, $A^{\mathcal{I}}(y) \geq r^{\mathcal{I}}(x,y)$ holds. Consider $x, y \in \Delta^{\mathcal{I}}$. Then, as $A \in \mathsf{ran}_{\mathcal{O}}(r)$

$$\begin{aligned}
r^{\mathcal{I}}(x,y) &= r^{\mathcal{J}}(x,y) \otimes (\bigsqcap_{B \in \mathsf{ran}_{\mathcal{O}}(r)} B)^{\mathcal{J}}(y) \\
&\leq r^{\mathcal{J}}(x,y) \otimes A^{\mathcal{J}}(y) \\
&\leq A^{\mathcal{J}}(y) \\
&= A^{\mathcal{I}}(y) \ ,
\end{aligned}$$

which concludes this and all cases.

Therefore, $\mathcal{I} \models \mathcal{O}$. As for $x \in \Delta^{\mathcal{I}}$, $A^{\mathcal{I}}(x) = A^{\mathcal{J}}(x)$, $B^{\mathcal{I}}(x) = B^{\mathcal{J}}(x)$ and $\mathcal{J} \not\models \langle A \sqsubseteq B, \alpha \rangle$, we have that $\mathcal{I} \not\models \langle A \sqsubseteq B, \alpha \rangle$ and, thus, $\mathcal{O} \not\models \langle A \sqsubseteq B, \alpha \rangle$, which concludes the proof. □

In the following, let us consider the following definition of *fuzzy conditional canonical interpretation*.

Definition 73 *(Fuzzy conditional canonical interpretation).* In G-$\mathcal{ELO}_{\bot}^{+}$, let $\mathcal{O}$ be an assertion free fuzzy ontology and assume that there is neither $\langle A\colon A \sqsubseteq \bot, \beta \rangle$ nor $\langle A\colon \{a\} \sqsubseteq \bot, \beta \rangle$ in $\mathsf{fclosure}(\mathcal{O})$, for all $\beta > 0$. Then, the *fuzzy conditional canonical interpretation* $\mathcal{I}$ w.r.t. $\mathcal{O}$ is defined as follows ($\max \emptyset = 0$).

$$\begin{aligned}
\Delta^{\mathcal{I}} &= \{x_C \mid A \rightsquigarrow C \in \mathsf{fclosure}(\mathcal{O}), \text{ and } \forall \beta > 0 \ \langle A\colon C \sqsubseteq \bot, \beta \rangle \notin \mathsf{fclosure}(\mathcal{O})\} \\
B^{\mathcal{I}}(x_C) &= \max\{\beta \mid \langle C \sqsubseteq B, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})\} \\
r^{\mathcal{I}}(x_{B_1}, x_{B_2}) &= \max\{\beta \mid \langle B_1 \sqsubseteq \exists r.B_2, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})\} \\
a^{\mathcal{I}} &= x_{\{a\}} \ .
\end{aligned}$$

Note that by rule ($\rightsquigarrow_0$), $A \rightsquigarrow A \in \mathsf{fclosure}(\mathcal{O})$ and, thus, by construction, $x_A \in \Delta^{\mathcal{I}} \neq \emptyset$. Moreover, by rule ($\rightsquigarrow_1$), $A \rightsquigarrow \{a\} \in \mathsf{fclosure}(\mathcal{O})$ and, thus, by construction, $x_{\{a\}} \in \Delta^{\mathcal{I}}$. Therefore, the interpretation $\mathcal{I}$ is well-defined.

Proposition 67. *In G-$\mathcal{ELO}_{\bot}^{+}$, let $\mathcal{O}$ be an assertion free fuzzy ontology and let $A, B \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. Then,*

1. $\mathsf{fclosure}(\mathcal{O})$ *can be computed in polynomial time w.r.t.* $|\mathcal{O}|$.
2. $\mathcal{O} \models \langle A \sqsubseteq B, \alpha \rangle$ *iff one of the following conditions holds:*
   (a) $\langle A\colon A \sqsubseteq B, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$ *with* $\beta \geq \alpha$*;*
   (b) $\langle A\colon A \sqsubseteq \bot, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$*;*
   (c) $\langle A\colon \{a\} \sqsubseteq \bot, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$.

Proof. Without loss of generality, we may assume that $\mathcal{O}$ is in normal form.

Computational complexity. The computational complexity result is inherited from Proposition 30 as (i) rules $(FOR_0) - (FOR_\circ)$ extend those in Fig. 5 without any non-polynomial blow-up; and (ii) the remaining rules $(\rightsquigarrow_0) - (\rightsquigarrow_3)$ can be applied in polynomial time and generate at most $|\mathcal{O}|^2$ new axioms.

Soundness. Soundness can be proven exactly as for Proposition 30 by showing that each rule is model preserving in the sense that if an interpretation $\mathcal{I}$ is a model of the antecedent of a rule then $\mathcal{I}$ is a model of the consequent of that rule.

Completeness. The proof follows the completeness of Proposition 30. So, assume that $\mathcal{O} \models \langle A \sqsubseteq B, \alpha\rangle$ holds. If there is $\langle A\colon A \sqsubseteq \bot, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ or $\langle A\colon \{a\} \sqsubseteq \bot, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ for some $\beta > 0$ then the $(\Rightarrow)$ direction holds trivially. Otherwise, assume that there is neither $\langle A\colon A \sqsubseteq \bot, \beta\rangle$ nor $\langle A\colon \{a\} \sqsubseteq \bot, \beta\rangle$ in $\mathsf{fclosure}(\mathcal{O})$, for all $\beta > 0$. Next, consider the fuzzy canonical interpretation $\mathcal{I}$ w.r.t. $\mathcal{O}$ constructed as in Definition 73, which is well-defined as $\{x_{\{a\}}, x_A\} \subseteq \Delta^{\mathcal{I}} \neq \emptyset$.

Now, it is not difficult to see that the analogue of Claims 1–4 hold here as well:

Claim 6. *For every $A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$ such that no $\langle A\colon A \sqsubseteq \bot, \beta\rangle$ is in $\mathsf{fclosure}(\mathcal{O})$, $x_A \in \Delta^{\mathcal{I}}$ and $A^{\mathcal{I}}(x_A) = 1$ hold.*

Proof. The proof is as for Claim 1. Consider $A \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$ such that no $\langle A\colon A \sqsubseteq \bot, \beta\rangle$ is in $\mathsf{fclosure}(\mathcal{O})$. By rule $(\rightsquigarrow_0)$, $A \rightsquigarrow A \in \mathsf{fclosure}(\mathcal{O})$ and, thus, by construction of $\mathcal{I}$, $x_A \in \Delta^{\mathcal{I}}$. Then by rule (FOR_0) we have that $\langle A\colon A \sqsubseteq A, 1\rangle \in \mathsf{fclosure}(\mathcal{O})$ and, thus, again by construction of $\mathcal{I}$, $A^{\mathcal{I}}(x_A) = 1$. □

Claim 7. *For each $x_C \in \Delta^{\mathcal{I}}$ and each concept D, $\langle A\colon C \sqsubseteq D, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ for some $\beta > 0$ implies $D^{\mathcal{I}}(x_C) \geq \beta$.*

Proof. The proof is the same as for Claim 2. For illustrative purposes we consider here only

Case $D = \exists r.B$: As axioms are in normal form, $C \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. Now, consider $\langle A\colon C \sqsubseteq \exists r.B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$. It cannot be the case that $\langle A\colon B \sqsubseteq \bot, \alpha\rangle \in \mathsf{fclosure}(\mathcal{O})$, as otherwise by rule $(FOR_\bot)$ we have $\langle A\colon C \sqsubseteq \bot, \beta \otimes \alpha\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \otimes \alpha > 0$, contrary to the assumption that $x_C \in \Delta^{\mathcal{I}}$. So, $x_B \in \Delta^{\mathcal{I}}$. By Claim 1, we have $B^{\mathcal{I}}(x_B) = 1$. Moreover, as $\langle A\colon C \sqsubseteq \exists r.B, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$, by construction of $r^{\mathcal{I}}$ we also have that $r^{\mathcal{I}}(x_A, x_B) \geq \beta$ and, thus, $(\exists r.B)^{\mathcal{I}}(x_A) \geq r^{\mathcal{I}}(x_A, x_B) \otimes B^{\mathcal{I}}(x_B) = r^{\mathcal{I}}(x_A, x_B) \geq \beta$. □

Claim 8. *Let C and D be two concepts such that $C \sqsubseteq D$ is in normal form. If $D^{\mathcal{I}}(x_C) \geq \alpha > 0$ then there is $\langle A\colon C \sqsubseteq D, \beta\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \geq \alpha$.*

Proof. The proof is the same as for Claim 3. For illustrative purposes we consider here only

Case $D = \exists r.B$: In this case $C \in \mathsf{N}_{\mathscr{C}}^{\mathcal{O}}$. As $D^{\mathcal{I}}(x_C) \geq \alpha$, by the definition of the $\exists$ constructor and witnessed property, there is $x_E \in \Delta^{\mathcal{I}}$ such that $\alpha \leq (\exists r.B)^{\mathcal{I}}(x_C) = r^{\mathcal{I}}(x_C, x_E) \otimes B^{\mathcal{I}}(x_E)$ and, thus, $r^{\mathcal{I}}(x_C, x_E) \geq \alpha$ and $B^{\mathcal{I}}(x_E) \geq \alpha$. By definition of $r^{\mathcal{I}}$ there is $\langle A\colon C \sqsubseteq \exists r.E, \beta_1\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_1 \geq \alpha$. By induction hypothesis on x_E and B there is $\langle A\colon E \sqsubseteq B, \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_2 \geq \alpha$. By the $(FOR_\exists)$ rule $\langle A\colon C \sqsubseteq \exists r.B, \beta_1 \otimes \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_1 \otimes \beta_2 \geq \alpha$. □

Claim 9. *$\mathcal{I}$ is a witnessed model of $\mathcal{O}$.*

Proof. The proof is the same as for Claim 4. For illustrative purposes we consider here only

Case $\langle D_1 \sqsubseteq D_2, \beta_2\rangle$. Consider $x_C \in \Delta^{\mathcal{I}}$ and assume $D_1^{\mathcal{I}}(x_C) = \alpha$. We have to show that $D_2^{\mathcal{I}}(x_C) \geq D_1^{\mathcal{I}}(x_C) \otimes \beta_2 = \alpha \otimes \beta_2$. Now, by Claim 8 there is $\langle A\colon C \sqsubseteq D_1, \beta_1\rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta_1 \geq \alpha$. By the $(FOR_{\sqsubseteq}^1)$ rule we have $\langle A\colon C \sqsubseteq D_2, \beta_1 \otimes \beta_2\rangle \in \mathsf{fclosure}(\mathcal{O})$. Eventually, by Claim 7, we have ${D_2}^{\mathcal{I}}(x_C) \geq \beta_1 \otimes \beta_2 \geq \alpha \otimes \beta_2$.

The argument supporting $\mathcal{I}$ being witnessed is the same as for Claim 4. □

We are ready now to complete the whole proof of the proposition. So, assume $\mathcal{O} \models \langle A \sqsubseteq B, \alpha \rangle$. Recall that we have assumed now that there is neither $\langle A\colon A \sqsubseteq \bot, \beta \rangle$ nor $\langle A\colon \{a\} \sqsubseteq \bot, \beta \rangle$ in $\mathsf{fclosure}(\mathcal{O})$, for all $\beta > 0$. So, we know that the fuzzy canonical interpretation $\mathcal{I}$ is well-founded and by Claim 9, $\mathcal{I} \models \mathcal{O}$. By Claim 6, $x_A \in \Delta^{\mathcal{I}}$ and $A^{\mathcal{I}}(x_A) = 1$. As $\mathcal{I} \models \langle A \sqsubseteq B, \alpha \rangle$, we have that $B^{\mathcal{I}}(x_A) \geq \alpha \otimes A^{\mathcal{I}}(x_A) = \alpha$. Therefore, by Claim 8, there is $\langle A\colon A \sqsubseteq B, \beta \rangle \in \mathsf{fclosure}(\mathcal{O})$ with $\beta \geq \alpha$, which concludes. □

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