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Generalized fuzzy rough description logics ☆

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A B S T R A C T

Description logics (DLs) are a family of logics for representing structured knowledge which have proved to be very useful as ontology languages. Classical DLs are not suitable to represent vague pieces of information. The attempts to achieve a solution have led to the birth of fuzzy DLs and rough DLs. In this work, we provide a simple solution to join these two formalisms and define a fuzzy rough DL. This logic is more general than other related approaches, including tight and loose fuzzy rough approximations and being independent of the fuzzy logic operators considered. We show the usefulness of our approach by presenting some uses case, and we also describe how to extend two reasoning algorithms for fuzzy DLs, which are implemented in the fuzzy DL reasoners FUZZYDL and DELOREAN.

1. Introduction

In the last years, the use of ontologies as formalisms for knowledge representation in many different application domains has grown significantly. Ontologies have been successfully used as part of expert and multiagent systems, as well as a core element in the Semantic Web, which proposes to extend the current web to give information a well-defined meaning [3].

An ontology is defined as an explicit and formal specification of a shared conceptualization [20], which means that ontologies represent the concepts and the relationships in a domain promoting interrelation with other models and automatic processing. Ontologies allow adding semantics to data, making knowledge maintenance, information integration, and reuse of components easier.

The current standard language for ontology creation is OWL 2 [12], which is the successor of the Web Ontology Language (OWL). OWL 2 has its logical foundation in *description logics* (DLs for short) [1]. We recall DLs are a family of logics for representing structured knowledge. Each logic is denoted by using a string of capital letters which identify the constructors of the logic and therefore its complexity. DLs have proved to be very useful as ontology languages [2]. For instance, OWL 2 is based on the DL $\mathcal{SROIQ}(\mathbf{D})$ [27].

Nevertheless, it is widely agreed that "classical" ontology languages are not appropriate to deal with imprecise and vague knowledge, which is inherent to several real world domains [48]. With the aim of managing vagueness in ontologies, several extension of DLs have been proposed, being possible to group them in two categories.

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- On the one hand, the combination with fuzzy logic [65] produces *fuzzy DLs* [38]. In the setting of fuzzy logics, the convention prescribing that a statement is either true or false is changed. This theory is based on the notion of fuzzy set. Rather than belonging to a set or not, every element of the domain partially belongs to a fuzzy set. Under this approach, vagueness is quantified and expressed using a degree of membership to a vague concept.
- On the other hand, the combination with rough set theory [45] produces *rough DLs*. Rough set theory offers a qualitative approach to model vagueness. Instead of providing a degree of membership, a vague concept is approximated by means of two sets: an upper and a lower approximation, using an indiscernibility relation between the elements of the domain. This approach is very useful when it is not possible to quantify the membership function of a vague concept.

Fuzzy logic and rough logic are complementary formalisms to manage vagueness and hence it is natural to combine them by means of *fuzzy rough sets* [13,46]. The main difference with respect to rough theory are the presence of fuzzy rough sets (instead of fuzzy sets) and fuzzy similarity relations (instead of indiscernibility relations).

In this article we follow this approach and extend a fuzzy DL with fuzzy rough sets. In particular, we present a fuzzy rough extension of the DL $\mathcal{SROIQ}(\mathbf{D})$, providing the theoretical grounding for fuzzy rough OWL 2.

This combination is useful in several domains of application. For instance, in e-commerce, it is possible to combine rough concepts such as "potential buyer" (an individual which is possibly interested in some product) with fuzzy concepts such as "cheap price" (which can be modeled with a trapezoidal membership function). Another example is medicine, which combines rough concepts such as "possible patient" (an individual affected by some of the symptoms of some disease, and hence suspected of being patient) with fuzzy concepts such as "high blood pressure".

Our contribution to the field of ontologies is the possibility to represent and reason with vague knowledge by means of both a quantitative and a qualitative approach. Our contribution to the field of fuzzy rough set theory is the possibility to take benefit of the advantages of using ontologies, such as making it easier the definition of a common and agreed vocabulary, interoperability, knowledge reuse, information integration, and maintenance.

Compared to other fuzzy rough DLs [28,29], our work is more general because (i) it allows different indiscernibility relations that can be represented using fuzzy similarity or fuzzy equivalence relations, (ii) the semantics of our logic is independent from the fuzzy logic considered, (iii) besides lower and upper rough approximations, it supports tight and loose rough approximations [13]. We will also present some practical use cases showing the benefits of using a fuzzy rough DL, and report our experiences of implementing the logic in two well-known fuzzy DL reasoners. As we will see, the integration is seamless, as already pointed in crisp (non-fuzzy) DLs [50], as the rough set component can be mapped into the fuzzy DL component, with the non-negligible advantage that current fuzzy DLs reasoners can be used with minimal adaption.

The remainder of this work is organized as follows. Section 2 overviews some necessary background on mathematical fuzzy logics, fuzzy rough set theory and the classical DL $\mathcal{SROIQ}(\mathbf{D})$. Section 3 presents the definition of a fuzzy rough extension of $\mathcal{SROIQ}(\mathbf{D})$, the logic behind OWL 2. Section 4 presents some use cases, and then Section 5 discusses two implementations of reasoning algorithms for two fragments of our logic, in the fuzzyDL and DeLorean systems. Next, Section 6 compares our approach with the related work. Finally, Section 7 sets out some conclusions and ideas for future research.

2. Preliminaries

This section provides some basic background. Section 2.1 refreshes some basic ideas in mathematical fuzzy logic. Then, Section 2.2 recalls rough set and fuzzy rough set theories. Finally, Section 2.3 considers the classical DL $\mathcal{SROIQ}(\mathbf{D})$.

2.1. Mathematical fuzzy logic

Fuzzy set theory and fuzzy logic were proposed by Zadeh [65] to manage imprecise and vague knowledge.

While in classical set theory elements either belong to a set or not, in fuzzy set theory elements can belong to a set to some degree. More formally, let X be a set of elements called the reference set. A *fuzzy set A* over a countable crisp set X is defined by a membership function $\mu_A(x)$, or simply $A(x)$, which assigns any $x \in X$ to a value in the interval $[0,1]$. As in the classical case, 0 means no-membership and 1 means full membership, but now a value between 0 and 1 represents the extent to which x can be considered an element of X. The set of fuzzy sets on X is denoted as $\mathfrak{F}(X)$.

In the setting of fuzzy logics, the convention prescribing that a statement is either true or false is changed. A more refined range is used for the function that represents the meaning of a statement. This is usual in natural language when words are modelled by fuzzy sets. For example, the compatibility of "tall" in the phrase "a tall man" with some individual of a given height is often graded: the man can be judged not quite tall, somewhat tall, rather tall, very tall, etc.

Changing the usual true/false convention leads to the new concept of *fuzzy statement*, whose compatibility with a given state of facts is a matter of degree. This degree of fit is called *degree of truth*, and can be measured on a *truth space* $\mathcal{S}$, usually $[0,1]$ (in that case we speak about *Mathematical fuzzy logic* [23]). In this paper, we consider fuzzy statements of the form $\phi \geqslant \alpha$ or $\phi \leqslant \beta$, where $\alpha, \beta \in [0,1]$ [22,23] and ϕ is a statement, which encode that the degree of truth of ϕ is *at least* α, resp. *at most* β. For example, $\mathsf{ripeTomato} \geqslant 0.9$ says that we have a rather ripe tomato (the degree of truth of $\mathsf{ripeTomato}$ is at least 0.9).

Fuzzy logics provide compositional calculi of degrees of truth. All crisp set operations are extended to fuzzy sets. The intersection, union, complement and implication set operations are performed in the fuzzy case by a t-norm function $\otimes$, a

Table 1
Fuzzy connectives of some outstanding fuzzy logics.

Connective	Łukasiewicz	Gödel	Product	Zadeh
$\alpha \otimes \beta$	$\max(\alpha+\beta-1,0)$	$\min(\alpha,\beta)$	$\alpha \cdot \beta$	$\min(\alpha,\beta)$
$\alpha \oplus \beta$	$\min(\alpha+\beta,1)$	$\max(\alpha,\beta)$	$\alpha+\beta-\alpha\cdot\beta$	$\max(\alpha,\beta)$
$\alpha \Rightarrow \beta$	$\min(1-\alpha+\beta,1)$	$\begin{cases} 1 & \text{if } \alpha \leqslant \beta \\ \beta & \text{otherwise} \end{cases}$	$\min(1,\beta/\alpha)$	$\max(1-\alpha,\beta)$
$\ominus\alpha$	$1-\alpha$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$\begin{cases} 1 & \text{if } \alpha = 0 \\ 0 & \text{otherwise} \end{cases}$	$1-\alpha$

Table 2
Some properties of some outstanding fuzzy logics.

Property	Łukasiewicz	Gödel	Product	Zadeh
$\alpha \otimes (\ominus\alpha) = 0$	+	+	+	–
$\alpha \oplus (\ominus\alpha) = 1$	+	–	–	–
$\alpha \otimes \alpha = \alpha$	–	+	–	+
$\alpha \oplus \alpha = \alpha$	–	+	–	+
$\ominus\ominus\alpha = \alpha$	+	–	–	+
$\alpha \Rightarrow \beta = \ominus\alpha \oplus \beta$	+	–	–	+
$\ominus(\alpha \Rightarrow \beta) = \alpha \otimes \ominus \beta$	+	–	–	+
$\ominus(\alpha \otimes \beta) = \ominus\alpha \oplus \ominus \beta$	+	+	+	+
$\ominus(\alpha \oplus \beta) = \ominus\alpha \otimes \ominus \beta$	+	+	+	+

t-conorm function $\oplus$, a negation function $\ominus$ and an implication function $\Rightarrow$, respectively[1] (see [23] for a formal definition of these functions and their properties). Several t-norms, t-conorms, implications, and negations have been given in the literature; some outstanding examples are shown in Table 1.

We will sometimes identify a particular fuzzy operator by adding a subscript to a fuzzy connective, denoting the fuzzy logic that it belongs to (typically, Zadeh, Gödel, Łukasiewicz, or Product). For instance, $\otimes_G$ denotes Gödel t-norm, whereas $\Rightarrow_Ł$ denoted Łukasiewicz implication. The implication $\alpha \Rightarrow_{KD} \beta = \max(1-\alpha,\beta)$ is called *Kleene–Dienes implication* in the fuzzy logic literature.

We will recall here some important properties of these functions that will be used in this paper. An *involutive* negation satisfies that $\ominus(\ominus\alpha) = \alpha$. For instance, Łukasiewicz negation is involutive, while Gödel negation is not. Usually, the implication function $\Rightarrow$ is defined as an *R-implication*, or the residuum of a left-continuous t-norm $\otimes$, that is, $\alpha \Rightarrow \beta = \sup\{\gamma | \alpha \otimes \gamma \leqslant \beta\}$. An *S-implication* is defined as $\alpha \Rightarrow \beta = \ominus\alpha \oplus \beta$ (in this case we say $\ominus$ is the negation associated to $\Rightarrow$). Łukasiewicz implication is both an R-implication and an S-implication. Gödel and product fuzzy logics have an R-implication, whereas Zadeh fuzzy logic has an S-implication.

In this paper, a quadruple composed by a t-norm, a t-conorm, an implication function and a negation function determines a *fuzzy logic* (usually called a family of fuzzy operators). The most important fuzzy logics are Łukasiewicz (denoted Ł), Gödel (denoted G), and Product logic (denoted Π), due to the fact that any continuous t-norm can be obtained as a combination of Łukasiewicz, Gödel, and Product t-norm [40]. The so-called "Zadeh logic" is subsumed by Łukasiewicz fuzzy logic, since every fuzzy operator of Zadeh logic can be simulated with the fuzzy operators of Łukasiewicz logic. In fact, let the subscripts Z and Ł denote that the fuzzy operator corresponds to the Zadeh fuzzy logic and to the Łukasiewicz fuzzy logic, respectively. It is easy to check that:

$$\ominus_Z\alpha = \ominus_Ł\alpha \qquad \alpha\otimes_Z\beta = \alpha\otimes_Ł(\alpha \Rightarrow_Ł \beta)$$
$$\alpha\oplus_Z\beta = (\alpha \Rightarrow_Ł \beta) \Rightarrow_Ł \beta \qquad \alpha \Rightarrow_Z \beta = (\ominus_Z\alpha) \oplus_Z \beta$$

Some salient properties of these four logics are shown in Table 2. For more properties, see especially [23,44]. Note also, that a fuzzy logic having all properties shown in Table 2, collapses to Boolean logic, i.e. the truth-set can be $\{0,1\}$ only.

Relations can also be extended to the fuzzy case. A (binary) *fuzzy relation* R over two countable crisp sets X and Y is a function $R: X \times Y \to [0,1]$. The *inverse* of R is the function $R^-: Y \times X \to [0,1]$ with membership function $R^-(y,x) = R(x,y)$, for every $x \in X$ and $y \in Y$.

Implication functions and t-norms are also used to define the degree of subsumption between fuzzy sets, the composition of fuzzy relations, and the transitivity of fuzzy relations.

Given two fuzzy sets A and B, A is *included* in B (denoted $A \subseteq B$) iff $\forall x \in X, A(x) \leqslant B(x)$. Note that $A \subseteq B$ evaluates to a value $\alpha \in \{0,1\}$. A generalization of this inclusion is the *degree of subsumption*, defined as $\inf_{x \in X}\{A(x) \Rightarrow B(x)\}$ for an implication function $\Rightarrow$, that takes values in $[0,1]$.

[1] Note that $\ominus$ is also used in the context of Łukasiewicz logic to denote the binary connective $\alpha \ominus \beta = \max\{0, \alpha-\beta\}$. However, in this paper we will use it as a unary negation function.

The *composition* of two fuzzy relations $R_1 : X \times Y \to [0,1]$ and $R_2 : Y \times Z \to [0,1]$ is defined as $(R_1 \circ R_2)(x,z) = \sup_{y \in Y} R_1(x,y) \otimes R_2(y,z)$. A fuzzy relation $R : X \times X$ is $\otimes$-*transitive* iff $\forall x, y, z \in X$, $R(x,y) \geqslant R(x,z) \otimes R(z,y)$.

We say that R is an *equivalence* relation if it is reflexive, symmetric, and transitive. Similarly, R is a *similarity* relation (also called a *tolerance* relation) if it is reflexive and symmetric.

A fuzzy interpretation $\mathcal{I}$ *satisfies* a fuzzy statement $\phi \geqslant \alpha$ (resp., $\phi \leqslant \beta$) or $\mathcal{I}$ is a *model* of $\phi \geqslant \alpha$ (resp., $\phi \leqslant \beta$), denoted $\mathcal{I} \models \phi \geqslant \alpha$ (resp., $\mathcal{I} \models \phi \leqslant \beta$), iff $\mathcal{I}(\phi) \geqslant \alpha$ (resp., $\mathcal{I}(\phi) \leqslant \beta$). The notions of satisfiability and logical consequence are defined in the standard way. We say that $\phi \geqslant \alpha$ is a *tight logical consequence* of a set of fuzzy statements $\mathcal{K}$ iff α is the infimum of $\mathcal{I}(\phi)$ subject to all models $\mathcal{I}$ of $\mathcal{K}$. We refer the reader to [23] for reasoning algorithms for fuzzy propositional and first-order logics.

2.2. Rough set and fuzzy rough set theories

The notion of rough set was introduced by Pawlak in 1982 [45]. The key idea in rough set theory is the approximation of a vague concept when there is only incomplete information about the concept. The available information includes some examples of elements that belong to the concept, and an indiscernibility equivalence (reflexive, symmetric, and transitive) or similarity (reflexive and symmetric) relation between elements of the domain.

Then, a vague concept is approximated by means of a pair of concepts: a sub-concept or lower approximation, and a super-concept or upper approximation, describing the sets of elements which definitely and possibly belong to the vague set, respectively, as Fig. 1 illustrates.

Given an indiscernibility relation s, the *lower approximation* is defined as:

$$(s \downarrow A) = \{x | \forall y : (x,y) \in s \to y \in A\}$$

Similarly, the *upper approximation* of a set A is defined as:

$$(s \uparrow A) = \{x | \exists y : (x,y) \in s \wedge y \in A\}$$

It is also common to denote the lower and upper approximations of a set A as $\underline{A}$ and $\overline{A}$, respectively.

A *rough set* is then defined as a pair of concepts: a lower approximation and an upper approximation of a vague concept.

A very natural extension is to consider a fuzzy similarity relation instead of an indiscernibility relation, which gives raise to *fuzzy rough sets*. In this paper, we follow the definition of fuzzy rough sets proposed in [13], which extends [46]. Other relevant works are [18,35,37,39,41,43,59,63].

Given a fuzzy similarity relation s, a t-norm $\otimes$ and an implication function $\Rightarrow$, the lower approximation $(s \downarrow A)$ and the upper approximation $(s \uparrow A)$ of a fuzzy set A are defined by the following membership functions:

$$(s \downarrow A)(x) = \inf_{y \in X} \{s(x,y) \Rightarrow A(y)\} \quad (1)$$

$$(s \uparrow A)(x) = \sup_{y \in X} \{s(x,y) \otimes A(y)\} \quad (2)$$

When the t-norm and the implication are not clear from the context, we will make them explicit in a subscript. For instance, $(s \downarrow A)_{Ł}$ denotes the lower approximation of A under Łukasiewicz implication, whereas $(s \downarrow A)_{KD}$ considers Kleene–Dienes implication. Analogously, $(s \uparrow A)_{Ł}$ denotes the upper approximation of A under Łukasiewicz t-norm, and $(s \uparrow A)_{G}$ under Gödel t-norm.

However, while in rough sets one element of the domain can only belong to one equivalence class, this is not true when we move into the fuzzy case. When taking into account the fact that an element can belong to several fuzzy similarity classes (with different degrees of truth), the notions of *tight* and *loose* approximation naturally appear [13]: a tight approximation considers all fuzzy similarity classes, whereas a loose approximation considers the best one among the similarity classes.

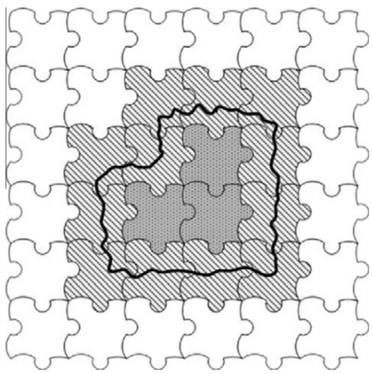

Fig. 1. Vague concept (bold line), upper approximation (light grey) and lower approximation (dark grey).

Given a fuzzy similarity relation s, a t-norm $\otimes$ and an implication function $\Rightarrow$, the tight lower approximation ($s \downarrow\downarrow A$), the loose lower approximation ($s \uparrow\downarrow A$), *the tight upper approximation* ($s \downarrow\uparrow A$), and the loose upper approximation ($s \uparrow\uparrow A$) of a fuzzy set A are defined by the following membership functions:

$$(s \downarrow\downarrow A)(x) = \inf_{z \in X} \left\{ s(x,z) \Rightarrow \inf_{y \in X} \{ s(z,y) \Rightarrow A(y) \} \right\} \quad (3)$$

$$(s \uparrow\downarrow A)(x) = \sup_{z \in X} \left\{ s(x,z) \otimes \inf_{y \in X} \{ s(z,y) \Rightarrow A(y) \} \right\} \quad (4)$$

$$(s \downarrow\uparrow A)(x) = \inf_{z \in X} \left\{ s(x,z) \Rightarrow \sup_{y \in X} \{ s(z,y) \otimes A(y) \} \right\} \quad (5)$$

$$(s \uparrow\uparrow A)(x) = \sup_{z \in X} \left\{ s(x,z) \otimes \sup_{y \in X} \{ s(z,y) \otimes A(y) \} \right\} \quad (6)$$

Again, if the t-norm and the implication are not clear from the context, we shall include them using a subscript. For instance, $(s \downarrow\downarrow A)_{Ł}$ denotes the tight lower approximation of A under Łukasiewicz implication.

2.3. The description logic $\mathcal{SROIQ}(\mathbf{D})$

The DL $\mathcal{SROIQ}(\mathbf{D})$ is the logical core of OWL 2. $\mathcal{SROIQ}(\mathbf{D})$ extends the standard DL $\mathcal{ALC}$ [51] with transitive roles ($\mathcal{ALC}$ plus transitive roles is called $\mathcal{S}$), complex role axioms ($\mathcal{R}$), nominals ($\mathcal{O}$), inverse roles ($\mathcal{I}$), qualified number restrictions ($\mathcal{Q}$), and concrete domains ($\mathbf{D}$).

2.3.1. Syntax

$\mathcal{SROIQ}(\mathbf{D})$ assumes three alphabets of symbols, for *concepts*, *roles* and *individuals*. *Abstract individuals* are elements of the abstract domain $\Delta^{\mathcal{I}}$, whereas *concrete individuals* are elements of a concrete domain $\Delta_{\mathbf{D}}$. Similarly, *abstract roles* relate two individuals, whereas *concrete roles* relate an individual and a concrete value.

A *concrete domain D* is a pair $\langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Delta_{\mathbf{D}}$ is a concrete interpretation domain and $\Phi_{\mathbf{D}}$ is a set of domain predicates $\mathbf{d}$ with a predefined arity n and an interpretation $\mathbf{d}_{\mathbf{D}} \subseteq {\Delta_{\mathbf{D}}}^n$. For simplicity we assume arity 1.

In DLs, complex concepts and roles can be built using different concept and role constructors. Before describing how to build them let us introduce some notation. C,D are (possibly complex) concepts, A is an atomic concept, R is a (possibly complex) role, R_A is an atomic abstract role, S is a simple role,[2] and universal role U), as shown in Table 3T is a concrete role, a, b are abstract individuals, v is a concrete individual, and n, m are natural numbers ($n \geqslant 0$, $m > 0$).

In $\mathcal{SROIQ}(\mathbf{D})$, the concepts and abstract roles can be built as shown in Table 3. Obviously, the inverse role is only defined for abstract roles.

Example 2.1. `Man` and `Woman` are atomic concepts. `hasChild` and `likes` are atomic roles. $\texttt{Man} \sqcap \geqslant 2\texttt{hasChild}.\texttt{Woman}$ is a complex concept representing a father with at least two daughters. $\exists\texttt{likes}.\texttt{Self}$ represents a narcisist.

A Knowledge Base (KB) comprises the intensional knowledge, i.e. general knowledge about the application domain (a Terminological Box or *TBox* $\mathcal{T}$ and a Role Box or *RBox* $\mathcal{R}$), and the extensional knowledge, i.e. particular knowledge about some specific situation (an Assertional Box or *ABox* $\mathcal{A}$ with statements about individuals).

An ABox consists of a finite set of *assertions* about individuals:

- *Concept assertions* $a{:}C$, meaning that individual a is an instance of C.
- *Role assertions* $(a,b){:}R$, meaning that (a,b) is an instance of R, and $(a,v){:}T$.
- *Negated role assertions* $(a,b):\neg R$, and $(a,v):\neg T$.
- *Inequality assertions* $a \neq b$.
- *Equality assertions* $a = b$.

A TBox consists of a finite set of *general concept inclusion* (*GCI*) axioms $C \sqsubseteq D$(C is more specific than D).

Let $w = R_1 R_2 \ldots R_n$ be a role chain (a finite string of roles not including the universal role U). An RBox consists of a finite set of role axioms:

- *Role inclusion axioms (RIAs)* $w \sqsubseteq R$ (role chain w is more specific than R), and $T_1 \sqsubseteq T_2$.
- *Transitive role axioms* $\texttt{trans}(R)$.
- *Disjoint role axioms* $\texttt{dis}(S_1,S_2)$, and $\texttt{dis}(T_1,T_2)$.
- *Reflexive role axioms* $\texttt{ref}(R)$.
- *Irreflexive role axioms* $\texttt{irr}(S)$.

[2] In order to prove decidability of the reasoning, some roles are required to be simple. Intuitively, simple roles cannot take part in cyclic role inclusion axioms. For a formal definition, we refer the reader to [27].

Table 3
Syntax and semantics of the description logic $\mathcal{SROIQ}(\mathbf{D})$.

Constructor	Syntax	Semantics
(Atomic concept)	A	$A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$
(Top concept)	$\top$	$\Delta^{\mathcal{I}}$
(Bottom concept)	$\bot$	$\emptyset$
(Concept conjunction)	$C \sqcap D$	$C^{\mathcal{I}} \cap D^{\mathcal{I}}$
(Concept disjunction)	$C \sqcup D$	$C^{\mathcal{I}} \cup D^{\mathcal{I}}$
(Concept negation)	$\neg C$	$\Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$
(Universal quantification)	$\forall R \cdot C$	$\{x \mid \forall y, (x,y) \notin R^{\mathcal{I}} \text{or } y \in C^{\mathcal{I}}\}$
(Existential quantification)	$\exists R \cdot C$	$\{x \mid \exists y, (x,y) \in R^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\}$
(Concrete universal quantification)	$\forall T \cdot \mathbf{d}$	$\{x \mid \forall v, (x,v) \notin T^{\mathcal{I}} \text{ or } v \in \mathrm{d_D}\}$
(Concrete existential quantification)	$\exists T \cdot \mathbf{d}$	$\{x \mid \exists v, (x,v) \in T^{\mathcal{I}} \text{ and } v \in \mathrm{d_D}\}$
(Nominal)	$\{a\}$	$\{a^{\mathcal{I}}\}$
(At-least number restriction)	$\geqslant nS \cdot C$	$\{x \mid \sharp\{y : (x,y) \in S^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \geqslant n\}$
(At-most number restriction)	$\leqslant nS \cdot C$	$\{x \mid \sharp\{y : (x,y) \in S^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}}\} \leqslant n\}$
(Concrete at-least number restriction)	$\geqslant nT \cdot \mathbf{d}$	$\{x \mid \sharp\{v : (x,v) \in T^{\mathcal{I}} \text{ and } v \in \mathrm{d_D}\} \geqslant n\}$
(Concrete at-most number restriction)	$\leqslant nT \cdot \mathbf{d}$	$\{x \mid \sharp\{v : (x,v) \in T^{\mathcal{I}} \text{ and } v \in \mathrm{d_D}\} \leqslant n\}$
(Local reflexivity)	$\exists S \cdot \mathtt{Self}$	$\{x \mid (x,x) \in S^{\mathcal{I}}\}$
(Atomic abstract role)	R_A	$R_A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$
(Concrete role)	T	$T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathrm{D}}$
(Inverse role)	R^-	$\{(y,x) \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (x,y) \in R^{\mathcal{I}}\}$
(Universal role)	U	$\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$

- *Symmetric role axioms* $\mathtt{sym}(R)$.
- *Asymmetric role axioms* $\mathtt{asy}(S)$.

Example 2.2. The concept assertion $\mathtt{paul}:\mathtt{Man}$ states that the individual Paul belongs to the class of men. The role assertion $(\mathtt{paul}, \mathtt{john}) : \neg\mathtt{hasChild}$ states that John is not the child of Paul. The GCI $\mathtt{Man} \sqsubseteq \mathtt{Human}$ states that all men are human. The RIA $\mathtt{owns}\ \mathtt{hasPart} \sqsubseteq \mathtt{owns}$ states the fact if somebody owns something, he also owns its components.

In order to guarantee the decidability of the logic, some additional syntactical restrictions are imposed in the form of the RIAs (see [27] for details) and in the use of simple roles, which is necessary in some concept constructors (local reflexivity, at-least and at-most number restrictions) and role axioms (disjoint, irreflexive and asymmetric role axioms).

2.3.2. Semantics

An interpretation $\mathcal{I}$ with respect to a concrete domain $\mathbf{D}$ is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) disjoint with $\Delta_{\mathbf{D}}$ and an interpretation function $\cdot^{\mathcal{I}}$ mapping:

- Every abstract individual a onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$.
- Every concrete individual v onto an element $v_{\mathbf{D}}$ of $\Delta_{\mathbf{D}}$.
- Every concept C onto a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$.
- Every abstract role R onto a relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.
- Every concrete role T onto a relation $T^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta_{\mathbf{D}}$.
- Every n-ary concrete predicate $\mathbf{d}$ onto the interpretation $d_{\mathbf{D}} \subseteq \Delta_{\mathbf{D}}^n$.

The interpretation is defined as shown in Table 3. Unique name assumption is not imposed, i.e. two nominals might refer to the same individual.

Let $\circ$ be the standard composition of relations. An interpretation $\mathcal{I}$ satisfies (is a model of):

- $a:C$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$,
- $(a,b):R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$,
- $(a,b) : \neg R$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \notin R^{\mathcal{I}}$,
- $(a,v):T$ iff $(a^{\mathcal{I}}, v_{\mathbf{D}}) \in T^{\mathcal{I}}$,
- $(a,v) : \neg T$ iff $(a^{\mathcal{I}}, v_{\mathbf{D}}) \notin T^{\mathcal{I}}$,
- $a \neq b$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$,
- $a = b$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$,
- $C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$,
- $R_1 \ldots R_n \sqsubseteq R$ iff $R_1^{\mathcal{I}} \circ \cdots \circ R_n^{\mathcal{I}} \subseteq R^{\mathcal{I}}$,
- $T_1 \sqsubseteq T_2$ iff $T_1^{\mathcal{I}} \subseteq T_2^{\mathcal{I}}$,

- $\texttt{trans}(R)$ iff $(x,y) \in R^{\mathcal{I}}$ and $(y,z) \in R^{\mathcal{I}}$ imply $(x,z) \in R^{\mathcal{I}},\ \forall x,y,z \in \Delta^{\mathcal{I}}$,
- $\texttt{dis}(S_1,S_2)$ iff $S_1^{\mathcal{I}} \cap S_2^{\mathcal{I}} = \emptyset$,
- $\texttt{dis}(T_1,T_2)$ iff $T_1^{\mathcal{I}} \cap T_2^{\mathcal{I}} = \emptyset$,
- $\texttt{ref}(R)$ iff $(x,x) \in R^{\mathcal{I}},\ \forall x \in \Delta^{\mathcal{I}}$,
- $\texttt{irr}(S)$ iff $(x,x) \notin S^{\mathcal{I}},\ \forall x \in \Delta^{\mathcal{I}}$,
- $\texttt{sym}(R)$ iff $(x,y) \in R^{\mathcal{I}}$ implies $(y,x) \in R^{\mathcal{I}},\ \forall x \in \Delta^{\mathcal{I}}$,
- $\texttt{asy}(S)$ iff $(x,y) \in S^{\mathcal{I}}$ implies $(y,x) \notin S^{\mathcal{I}},\ \forall x \in \Delta^{\mathcal{I}}$,
- A Knowledge Base $K = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}, \mathcal{T}$ and $\mathcal{R}$.

A DL not only stores axioms and assertions, but also offers some reasoning services, such as KB satisfiability, concept satisfiability or subsumption. However, if a DL is closed under negation, most of the basic reasoning tasks are reducible to KB satisfiability [49], so it is usually the only task considered.

3. Fuzzy rough $\mathcal{SROIQ}(\mathbf{D})$

In this section we define a fuzzy rough extension of the DL $\mathcal{SROIQ}(\mathbf{D})$ [27,31] where concepts denote fuzzy (or fuzzy rough) sets of individuals and roles denote fuzzy binary relations. Axioms are also extended to the fuzzy case and some of them hold to a degree. Our logic extends fuzzy $\mathcal{SROIQ}(\mathbf{D})$ [5,6,9] with some fuzzy rough constructors.

This section is organized as follows. The syntax and the semantics of the logic are presented in Sections 3.1 and 3.2, respectively. Section 3.3 defines the main reasoning problems. Finally, Section 3.4 collects some outstanding logical properties.

3.1. Syntax

To begin with, we will introduce two important elements of our logic: fuzzy concrete domains and fuzzy modifiers.

Fuzzy concrete domains A *fuzzy concrete domain* [55] $\mathbf{D}$ is a pair $\langle \Delta_{\mathbf{D}}, \Phi_{\mathbf{D}} \rangle$, where $\Delta_{\mathbf{D}}$ is a concrete interpretation domain, and $\Phi_{\mathbf{D}}$ is a set of fuzzy concrete predicates $\mathbf{d}$ with an arity n and an interpretation $\mathbf{d}_{\mathbf{D}} : \Delta_{\mathbf{D}}{}^n \to [0,1]$, which is an n-ary fuzzy relation over $\Delta_{\mathbf{D}}$.

As fuzzy concrete predicates, we allow the following functions defined over $[k_1, k_2] \subseteq \mathbb{Q}^+ \cup \{0\}$: trapezoidal membership function (Fig. 2(a)), triangular (Fig. 2(b)), left-shoulder function (Fig. 2(c)) and right-shoulder function (Fig. 2(d)) [55]. For backwards compatibility, we also allow crisp intervals (Fig. 2(e)). For instance, we may define $\mathsf{Young} : \mathbb{N} \to [0,1]$, denoting the degree of a person being young, as $\mathsf{Young}(x) = \texttt{left}(10,30)$. Formally:

$$
\begin{array}{lrcr}
\mathrm{d} \to & \texttt{crisp}(a,b) & | & \text{(crisp interval)}\\
 & \texttt{left}(a,b) & | & \text{(fuzzy left-shoulder function)}\\
 & \texttt{right}(a,b) & | & \text{(fuzzy right-shoulder function)}\\
 & \texttt{triangular}(a,b,c) & | & \text{(fuzzy triangular function)}\\
 & \texttt{trapezoidal}(a,b,c,d) & & \text{(fuzzy trapezoidal function)}
\end{array}
$$

3.1.1. Fuzzy modifiers

A *fuzzy modifier mod* is a function $f_{mod} : [0,1] \to [0,1]$ which applies to a fuzzy set to change its membership function. We will allow modifiers defined in terms of linear hedges (Fig. 2(f))[3] and triangular functions (Fig. 2(b)) [55]. In linear modifiers, we assume that $a = c/(c+1)$, $b = 1/(c+1)$. For instance, $\mathsf{very}(x) = \texttt{linear}(0.8)$. Formally:

$$
\begin{array}{lrcr}
mod \to & \texttt{linear}(c) & | & \text{(fuzzy linear modifier)}\\
 & \texttt{triangular}(a,b,c) & & \text{(fuzzy triangular modifier)}
\end{array}
$$

3.1.2. Alphabets of symbols

As for its crisp counterpart, fuzzy $\mathcal{SROIQ}(\mathrm{D})$ assumes three alphabets of symbols, for *fuzzy concepts*, *fuzzy roles* and *individuals*. *Abstract individuals* are elements of the abstract domain $\Delta^{\mathcal{I}}$, whereas *concrete individuals* are elements of a concrete domain $\Delta_{\mathbf{D}}$. Similarly, fuzzy *abstract roles* relate two individuals, whereas fuzzy *concrete roles* relate an individual and a concrete value.

3.1.3. Notation

Let us introduce some notation that will be used in the rest of the paper. C, D are (possibly complex) fuzzy concepts, A is an atomic fuzzy concept, R is a (possibly complex) abstract fuzzy role, R_A is an atomic fuzzy role, S is a simple fuzzy role, T is a concrete fuzzy role, a, b are abstract individuals, v is a concrete individual, $\mathbf{d}$ are fuzzy concrete predicates, n, m are natural

[3] Linear hedges are actually piecewise linear hedges, but we use the former name as it is usual in the fuzzy DL literature.

numbers ($n \geqslant 0, m > 0$), s_i is a fuzzy similarity relation, *mod* is a fuzzy modifier, $\bowtie \in \{\geqslant, <, \leqslant, >\}$, and $\alpha \in (0,1]$. Finally, $w = R_1 R_2 \ldots R_m$ is a role chain.

3.1.4. Fuzzy roles

The roles of the language can be built inductively according to the following rule:

$$
\begin{array}{lrcl}
R \rightarrow & R_A & | & (R1) \\
 & T & | & (R2) \\
 & R_{-} & | & (R3) \\
 & U & | & (R4) \\
 & mod(\mathrm{R}) & | & (R5) \\
 & [R \geqslant \alpha] & & (R6)
\end{array}
$$

Here, R^{-} denotes the inverse role of R which, as in the classical case, is only defined for abstract roles.

3.1.5. Fuzzy concepts

The concepts of the language can be built inductively as:

$$
\begin{array}{lrcl}
C, D \rightarrow & A & | & (C1) \\
 & \top & | & (C2) \\
 & \bot & | & (C3) \\
 & C \sqcap D & | & (C4) \\
 & C \sqcup D & | & (C5) \\
 & \neg C & | & (C6) \\
 & \forall R \cdot C & | & (C7) \\
 & \exists R \cdot C & | & (C8) \\
 & \forall T \cdot \mathsf{d} & | & (C9) \\
 & \exists T \cdot \mathsf{d} & | & (C10) \\
 & \{\alpha/a\} & | & (C11) \\
 & (\geqslant mS \cdot C) & | & (C12) \\
 & (\leqslant nS \cdot C) & | & (C13) \\
 & (\geqslant mT \cdot \mathsf{d}) & | & (C14) \\
 & (\leqslant nT \cdot \mathsf{d}) & | & (C15) \\
 & \exists S \cdot \mathtt{Self} & | & (C16) \\
 & C \rightarrow D & | & (C17) \\
 & \alpha_1 C_1 + \cdots + \alpha_m C_m & | & (C18) \\
 & mod(C) & | & (C19) \\
 & [C \geqslant \alpha] & | & (C20) \\
 & [C \leqslant \alpha] & | & (C21) \\
 & (s_i \downarrow\downarrow C) & | & (C22) \\
 & (s_i \uparrow\downarrow C) & | & (C23) \\
 & (s_i \downarrow C) & | & (C24) \\
 & (s_i \downarrow\uparrow C) & | & (C25) \\
 & (s_i \uparrow\uparrow C) & | & (C26) \\
 & (s_i \uparrow C) & & (C27)
\end{array}
$$

Example 3.1. Concept $\mathsf{Human} \sqcap \exists \mathsf{hasAge}.\mathtt{left}(10,30)$ denotes the set of young humans, with an age given by $\mathtt{left}(10,30)$. If $\mathtt{linear}(4)$ represents the modifier very, $Human \sqcap \mathtt{linear}(4)(\exists \mathsf{hasAge}.\mathtt{left}(10,30))$ denotes the set of *very* young humans. Finally, $(s \uparrow \mathsf{Buyer})$ represents the concept of potential buyer.

Concept constructors C1–C16 correspond to the concept constructors of crisp $\mathcal{SROIQ}(\mathsf{D})$. The only difference here is the presence of fuzzy nominals of the form $\{\alpha/a\}$ [5].

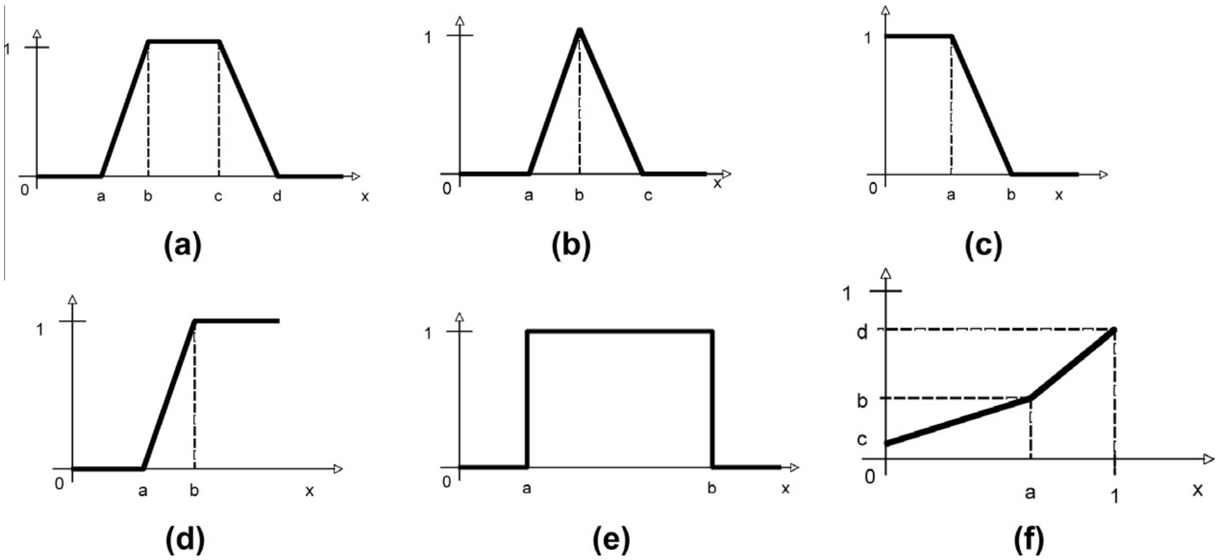

Fig. 2. (a) Trapezoidal function; (b) Triangular function; (c) L-function; (d) R-function; (e) Crisp interval; (f) Linear function.

Concept constructors C17–C21 are usual in the setting of fuzzy DLs, namely implication concepts (C17), weighted sum concepts (C18), modified concepts (C19), and cut concepts (C20–C21). In weighted sum concepts we assume that $\sum_{i=1,\ldots,m}\alpha_i \leqslant 1$.

Finally, concept constructors C22–C27 deal with fuzzy rough sets. Note that we allow to use k different fuzzy similarity relations s_i, $1 \leqslant i \leqslant k$. Note also that in a preliminary version of this paper we used fuzzy equivalence relations instead of fuzzy similarity relations [10]. This change would be justified in Section 3.4.

An easy extension which is not explained in detail here for the sake of clarity, is the addition of a superscript denoting the fuzzy logic that is used in the semantics of the constructor (typically, Zadeh, Gödel, Łukasiewicz, or Product). For instance, $\sqcap_G$ denotes a conjunction which is interpreted as the minimum (Gödel t-norm). Also, $(s \uparrow C)_G$ is an upper approximation under Gödel t-norm. Clearly, this only makes sense for those constructors which have negation, t-norm, t-conorm, or implication functions involved in their semantics, as defined in Section 3.2.[4]

3.1.6. Fuzzy Knowledge Base

A *Fuzzy Knowledge Base* (KB) contains a finite set of axioms of one of the following types:

- (A1) *Fuzzy concept assertion* of the form $\langle a : C \bowtie \alpha \rangle$.
- (A2) *Fuzzy role assertion* of the form $\langle (a,b) : R \bowtie \alpha \rangle$.
- (A3) *Fuzzy role assertion* of the form $\langle (a,b) : \neg R \bowtie \alpha \rangle$.
- (A4) *Fuzzy role assertion* of the form $\langle (a,v) : T \bowtie \alpha \rangle$.
- (A5) *Fuzzy role assertion* of the form $\langle (a,v) : \neg T \bowtie \alpha \rangle$.
- (A6) *Inequality assertion* $\langle a \neq b \rangle$.
- (A7) *Equality assertion* $\langle a = b \rangle$.
- (A8) Fuzzy General Concept Inclusion (*fuzzy GCI*) of the forms $\langle C \sqsubseteq D \geqslant \alpha \rangle$, or $\langle C \sqsubseteq D > \alpha \rangle$.
- (A9) Fuzzy Role Inclusion Axiom (*fuzzy RIA*) of the forms $\langle w \sqsubseteq R \geqslant \alpha \rangle$, or $\langle w \sqsubseteq R > \alpha \rangle$.
- (A10) Fuzzy Role Inclusion Axiom or *fuzzy RIA* of the forms $\langle T_1 \sqsubseteq T_2 \geqslant \alpha \rangle$, or $\langle T_1 \sqsubseteq T_2 > \alpha \rangle$.
- (A11) *Transitive* role axiom $\mathtt{trans}(R)$.
- (A12) *Disjoint* abstract role axiom $\mathtt{dis}(S_1, S_2)$.
- (A13) *Disjoint* concrete role axiom $\mathtt{dis}(T_1, T_2)$.
- (A14) *Reflexive* role axiom $\mathtt{ref}(R)$.
- (A15) *Irreflexive* role axiom $\mathtt{irr}(S)$.
- (A16) *Symmetric* role axiom $\mathtt{sym}(R)$.
- (A17) *Asymmetric* role axiom $\mathtt{asy}(S)$.

[4] This notation mixes in some way syntax with semantics, but makes it possible to combine fuzzy logical operators belonging to different fuzzy logics, according to the logical properties that the particular application domain requires.

Example 3.2. The fuzzy concept assertion $\langle \mathsf{paul} : \mathsf{Tall} \geqslant 0.5 \rangle$ states that Paul is tall with at least degree 0.5. The fuzzy RIA $\langle \mathsf{isFriendOf}\ \mathsf{isFriendOf} \sqsubseteq \mathsf{isFriendOf} \geqslant 0.75 \rangle$ states that the friends of my friends can also be considered as my friends with at least degree 0.75.

A fuzzy KB is organized in a *fuzzy ABox* $\mathcal{A}$ with axioms (A1)–(A7), a *fuzzy TBox* $\mathcal{T}$ with axioms (A8), and a *fuzzy RBox* $\mathcal{R}$ with axioms (A9)–(A17).

The pair of axioms $\langle \tau \geqslant \alpha \rangle$ and $\langle \tau \leqslant \alpha \rangle$ is abbreviated as $\langle \tau = \alpha \rangle$ [26]. Also, an axiom of the form $\langle \tau \geqslant 1 \rangle$ is usually abbreviated to simply τ.

Notice that fuzzy GCIs or RIAs of the forms $\langle \tau \leqslant \alpha \rangle$ and $\langle \tau < \alpha \rangle$ are not allowed, because they do not have an equivalence in crisp $\mathcal{SROIQ}(\mathsf{D})$.

Similarly as in the crisp case, there are some restrictions to guarantee the decidability of the logic: some roles are assumed to be simple in some concept constructors and role axioms, and there are some additional restrictions in the form of RIAs [6].

3.2. Semantics

Fuzzy interpretation A fuzzy interpretation $\mathcal{I}$ with respect to a fuzzy concrete domain **D** is a pair $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ (the interpretation domain) disjoint with $\Delta_{\mathbf{D}}$ and a fuzzy interpretation function $\cdot^{\mathcal{I}}$ mapping:

- A fuzzy *abstract individual a* onto an element $a^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$.
- A fuzzy *concrete individual* v onto an element $v_{\mathbf{D}}$ of $\Delta_{\mathbf{D}}$.
- A fuzzy (possibly rough) *concept C* onto a function $C^{\mathcal{I}} : \Delta^{\mathcal{I}} \to [0,1]$.
- A fuzzy *abstract role R* onto a function $R^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$.
- A fuzzy *concrete role T* onto a function $T^{\mathcal{I}} : \Delta^{\mathcal{I}} \times \Delta_{\mathsf{D}} \to [0,1]$.
- An *n*-ary fuzzy *concrete predicate* **d** onto the fuzzy relation $\mathbf{d}_{\mathbf{D}} : {\Delta_{\mathbf{D}}}^n \to [0,1]$.
- A fuzzy *modifier mod* onto a function $f_{mod} : [0,1] \to [0,1]$.

$C^{\mathcal{I}}$ (resp. $R^{\mathcal{I}}$) denotes the membership function of the fuzzy (possibly rough) concept *C* (resp. fuzzy role *R*) w.r.t. $\mathcal{I} \cdot C^{\mathcal{I}}(a)$ (resp. $R^{\mathcal{I}}(a,b)$) gives us to what extent the individual *a* can be considered as an element of *C* (resp. to what extent (*a*,*b*) can be considered as an element of *R*) under the fuzzy interpretation $\mathcal{I}$.

Given a t-norm $\otimes$, a t-conorm $\oplus$, a negation function $\ominus$ and an implication function $\Rightarrow$, the fuzzy interpretation function is extended to complex concepts and roles as follows:

$$\top^{\mathcal{I}}(x) = 1$$
$$\bot^{\mathcal{I}}(x) = 0$$
$$(C \sqcap D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \otimes D^{\mathcal{I}}(x)$$
$$(C \sqcup D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \oplus D^{\mathcal{I}}(x)$$
$$(\neg C)^{\mathcal{I}}(x) = \ominus C^{\mathcal{I}}(x)$$
$$(\forall R.C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)\}$$
$$(\exists R.C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}}\{R^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y)\}$$
$$(\forall T \cdot \mathsf{d})^{\mathcal{I}}(x) = \inf_{v \in \Delta_{\mathsf{D}}}\{T^{\mathcal{I}}(x, v) \Rightarrow \mathsf{d}_{\mathsf{D}}(v)\}$$
$$(\exists T \cdot \mathsf{d})^{\mathcal{I}}(x) = \sup_{v \in \Delta_{\mathsf{D}}}\{T^{\mathcal{I}}(x, v) \otimes \mathsf{d}_{\mathsf{D}}(v)\}$$
$$\{\alpha/a\}^{\mathcal{I}}(x) = \alpha \text{ if } x = a^{\mathcal{I}}, 0 \text{ otherwise}$$
$$(\geqslant m\ S.C)^{\mathcal{I}}(x) = \sup_{y_1,\ldots,y_m \in \Delta^{\mathcal{I}}}\left(\min_{i=1}^{m}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}\right) \otimes (\otimes_{1 \leqslant j<k \leqslant m}\{y_j \neq y_k\})$$
$$(\leqslant n\ S.C)^{\mathcal{I}}(x) = \inf_{y_1,\ldots,y_{n+1} \in \Delta^{\mathcal{I}}}\left(\min_{i=1}^{n+1}\{S^{\mathcal{I}}(x,y_i) \otimes C^{\mathcal{I}}(y_i)\}\right) \Rightarrow (\oplus_{1 \leqslant j<k \leqslant n+1}\{y_j = y_k\})$$
$$(\geqslant m\ T \cdot \mathsf{d})^{\mathcal{I}}(x) = \sup_{v_1,\ldots,v_m \in \Delta_{\mathsf{D}}}(\min_{i=1}^{m}\{T^{\mathcal{I}}(x, v_i) \otimes \mathsf{d}_{\mathsf{D}}(v_i)\}) \otimes (\otimes_{j<k}\{v_j \neq v_k\})$$
$$(\leqslant n\ T \cdot \mathsf{d})^{\mathcal{I}}(x) = \inf_{v_1,\ldots,v_{n+1} \in \Delta_{\mathsf{D}}}\left(\min_{i=1}^{n+1}\{T^{\mathcal{I}}(x, v_i) \otimes \mathsf{d}_{\mathsf{D}}(v_i)\}\right) \Rightarrow (\oplus_{j<k}\{v_j = v_k\})$$
$$(\exists S \cdot \mathtt{Self})^{\mathcal{I}}(x) = S^{\mathcal{I}}(x,x)$$
$$(C \to D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$$
$$(\alpha_1 C_1 + \cdots + \alpha_m C_m)^{\mathcal{I}}(x) = \alpha_1 C_1^{\mathcal{I}}(x) + \cdots + \alpha_m C_m^{\mathcal{I}}(x)$$
$$(mod(C))^{\mathcal{I}}(x) = f_{mod}(C^{\mathcal{I}}(x))$$
$$([C \geqslant \alpha])^{\mathcal{I}}(x) = 1 \text{ if } C^{\mathcal{I}}(x) \geqslant \alpha, 0 \text{ otherwise}$$
$$([C \leqslant \alpha])^{\mathcal{I}}(x) = 1 \text{ if } C^{\mathcal{I}}(x) \leqslant \alpha, 0 \text{ otherwise}$$
$$(s_i \downarrow\downarrow C)^{\mathcal{I}}(x) = \inf_{z \in X}\{s_i^{\mathcal{I}}(x,z) \Rightarrow \inf_{y \in \Delta^{\mathcal{I}}}\{s_i^{\mathcal{I}}(z,y) \Rightarrow C^{\mathcal{I}}(y)\}\}$$
$$(s_i \uparrow\downarrow C)^{\mathcal{I}}(x) = \sup_{z \in X}\{s_i^{\mathcal{I}}(x,z) \otimes \inf_{y \in \Delta^{\mathcal{I}}}\{s_i^{\mathcal{I}}(z,y) \Rightarrow C^{\mathcal{I}}(y)\}\}$$

$$(s_i \downarrow C)^{\mathcal{I}}(x) = \inf_{y \in \Delta^{\mathcal{I}}} \left\{ s_i^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y) \right\}$$
$$(s_i \downarrow\uparrow C)^{\mathcal{I}}(x) = \inf_{z \in X} \left\{ s_i^{\mathcal{I}}(x,z) \Rightarrow \sup_{y \in \Delta^{\mathcal{I}}} \left\{ s_i^{\mathcal{I}}(z,y) \otimes C^{\mathcal{I}}(y) \right\} \right\}$$
$$(s_i \uparrow\uparrow C)^{\mathcal{I}}(x) = \sup_{z \in X} \left\{ s_i^{\mathcal{I}}(x,z) \otimes \sup_{y \in \Delta^{\mathcal{I}}} \left\{ s_i^{\mathcal{I}}(z,y) \otimes C^{\mathcal{I}}(y) \right\} \right\}$$
$$(s_i \uparrow C)^{\mathcal{I}}(x) = \sup_{y \in \Delta^{\mathcal{I}}} \left\{ s_i^{\mathcal{I}}(x,y) \otimes C^{\mathcal{I}}(y) \right\}$$
$$(R^-)^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$$
$$U^{\mathcal{I}}(x,y) = 1$$
$$(mod(R))^{\mathcal{I}}(x,y) = f_{mod}(R^{\mathcal{I}}(x,y))$$
$$([R \geqslant \alpha])^{\mathcal{I}}(x,y) = 1 \text{ if } R^{\mathcal{I}}(x,y) \geqslant \alpha, \text{ 0 otherwise}$$

The fuzzy interpretation function is extended to fuzzy axioms as follows:

$$(a : C)^{\mathcal{I}} = C^{\mathcal{I}}(a^{\mathcal{I}})$$
$$((a,b) : R)^{\mathcal{I}} = R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$$
$$((a,b) : \neg R)^{\mathcal{I}} = \ominus R^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$$
$$(a, v) : T = T^{\mathcal{I}}(a^{\mathcal{I}}, v_{\mathsf{D}})$$
$$(a, v) : \neg T = \ominus T^{\mathcal{I}}(a^{\mathcal{I}}, v_{\mathsf{D}})$$
$$(C \sqsubseteq D)^{\mathcal{I}} = \inf_{x \in \Delta^{\mathcal{I}}} \{ C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \}$$
$$(R_1 \dots R_n \sqsubseteq R)^{\mathcal{I}} = \inf_{x_1, x_{n+1} \in \Delta^{\mathcal{I}}} \{ \sup_{x_2 \dots x_n \in \Delta^{\mathcal{I}}} \{ (R_1^{\mathcal{I}}(x_1,x_2) \otimes \cdots \otimes R_n^{\mathcal{I}}(x_n,x_{n+1})) \Rightarrow R^{\mathcal{I}}(x_1,x_{n+1}) \} \}$$
$$(T_1 \sqsubseteq T_2)^{\mathcal{I}} = \inf_{x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathsf{D}}} \{ T_1^{\mathcal{I}}(x, v) \Rightarrow T_2^{\mathcal{I}}(x, v) \}$$

Let $\phi \in \{a : C, (a,b) : R, (a,b) : \neg R, (a, v) : T, (a, v) : \neg T\}$ and $\psi \in \{C \sqsubseteq D, R_1 \dots R_m \sqsubseteq R, T_1 \sqsubseteq T_2\}$. A fuzzy interpretation $\mathcal{I}$ satisfies (is a model of):

- $\langle \phi \bowtie \gamma \rangle$ iff $\phi^{\mathcal{I}} \bowtie \gamma$.
- $\langle a \neq b \rangle$ iff $a^{\mathcal{I}} \neq b^{\mathcal{I}}$.
- $\langle a = b \rangle$ iff $a^{\mathcal{I}} = b^{\mathcal{I}}$.
- $\langle \psi \rhd \gamma \rangle$ iff $\psi^{\mathcal{I}} \rhd \gamma$.
- $\mathtt{trans}(R)$ iff $\forall x,y,z \in \Delta^{\mathcal{I}},\ R^{\mathcal{I}}(x,z) \otimes R^{\mathcal{I}}(z,y) \leqslant R^{\mathcal{I}}(x,y)$.
- $\mathtt{dis}(S_1,S_2)$ iff $\forall x,y \in \Delta^{\mathcal{I}}, S_1^{\mathcal{I}}(x,y) = 0$ or $S_2^{\mathcal{I}}(x,y) = 0$.
- $\mathtt{dis}(T_1,T_2)$ iff $\forall x \in \Delta^{\mathcal{I}}, v \in \Delta_{\mathsf{D}}, T_1^{\mathcal{I}}(x, v) = 0$ or $T_2^{\mathcal{I}}(x, v) = 0$.
- $\mathtt{ref}(R)$ iff $\forall x \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,x) = 1$.
- $\mathtt{irr}(S)$ iff $\forall x \in \Delta^{\mathcal{I}}, S^{\mathcal{I}}(x,x) = 0$.
- $\mathtt{sym}(R)$ iff $\forall x,y \in \Delta^{\mathcal{I}}, R^{\mathcal{I}}(x,y) = R^{\mathcal{I}}(y,x)$.
- $\mathtt{asy}(S)$ iff $\forall x,y \in \Delta^{\mathcal{I}}$, if $S^{\mathcal{I}}(x,y) > 0$ then $S^{\mathcal{I}}(y,x) = 0$.
- A fuzzy KB $\mathcal{K} = \langle \mathcal{A}, \mathcal{T}, \mathcal{R} \rangle$ iff it satisfies each element in $\mathcal{A}, \mathcal{T}$ and $\mathcal{R}$.

Notice that individual assertions are considered to be crisp, since the equality and inequality of individuals have always been considered crisp in the fuzzy DL literature [57,52].

3.3. Reasoning tasks

The reasoning tasks of fuzzy rough $\mathcal{SROIQ}(\mathsf{D})$ are the same as in fuzzy $\mathcal{SROIQ}(\mathsf{D})$ [5].

- *Fuzzy KB satisfiability*. Given a fuzzy KB $\mathcal{K}$, this problem consists on checking the existence of a fuzzy interpretation satisfying $\mathcal{K}$. Usually, it is the only reasoning task considered, since, as it happens in the crisp case, most inference problems can be reduced to it [54].
- *Fuzzy (possibly rough) concept satisfiability*. A fuzzy (possibly rough) concept C is α-satisfiable w.r.t. a fuzzy KB $\mathcal{K}$ iff there can exist models of $\mathcal{K}$ where C has instances with degree α, i.e. $C^{\mathcal{I}} \geqslant \alpha$.
- *Fuzzy entailment*: A KB $\mathcal{K}$ entails a fuzzy axiom τ of the forms (A1)–(A5) and (A8), denoted $\mathcal{K}$ models τ iff every model of $\mathcal{K}$ satisfies τ.
- *Fuzzy (possibly rough) concept subsumption*: A fuzzy (possibly rough) concept C is α-subsumed by a fuzzy (possibly rough) concept D w.r.t. a fuzzy KB $\mathcal{K}$ iff $\mathcal{K}$ entails the fuzzy GCI $\langle C \sqsubseteq D \geqslant \alpha \rangle$.
- *Greatest lower bound*. The greatest lower bound (glb) of an axiom $\phi \in \{a : C, (a,b) : R, (a,b) : \neg R, (a, v) : T, (a, v) : \neg T, C \sqsubseteq D\}$ w.r.t. a fuzzy KB $\mathcal{K}$ is defined as the $glb(\mathcal{K}, \phi) = \sup\{\alpha : \mathcal{K} \models \langle \phi \geqslant \alpha \rangle\}$.

3.4. Some properties

It can be easily shown that fuzzy rough $\mathcal{SROIQ}(\mathsf{D})$ is a sound extension of rough $\mathcal{SROIQ}(\mathsf{D})$, in the sense that fuzzy interpretations coincide with rough interpretations if we restrict the degrees of truth to {0, 1}.

Let s be a fuzzy similarity relation. The following properties derive directly from those of fuzzy rough sets [13]:

$$(s \downarrow\downarrow C) \sqsubseteq (s \downarrow C) \sqsubseteq C \sqsubseteq (s \uparrow C) \sqsubseteq (s \uparrow\uparrow C) \tag{7}$$
$$(s \downarrow C) \sqsubseteq (s \uparrow\downarrow C) \sqsubseteq (s \uparrow C) \tag{8}$$
$$(s \downarrow C) \sqsubseteq (s \downarrow\uparrow C) \sqsubseteq (s \uparrow C) \tag{9}$$
$$C \sqsubseteq D \text{ implies } (s \downarrow\downarrow C) \sqsubseteq (s \downarrow\downarrow D) \tag{10}$$
$$C \sqsubseteq D \text{ implies} (s \uparrow\downarrow C) \sqsubseteq (s \uparrow\downarrow D) \tag{11}$$
$$C \sqsubseteq D \text{ implies} (s \downarrow C) \sqsubseteq (s \downarrow D) \tag{12}$$
$$C \sqsubseteq D \text{ implies} (s \downarrow\uparrow C) \sqsubseteq (s \downarrow\uparrow D) \tag{13}$$
$$C \sqsubseteq D \text{ implies} (s \uparrow\uparrow C) \sqsubseteq (s \uparrow\uparrow D) \tag{14}$$
$$C \sqsubseteq D \text{ implies} (s \uparrow C) \sqsubseteq (s \uparrow D) \tag{15}$$
$$(s \downarrow\downarrow C \sqcap D) \equiv (s \downarrow\downarrow C) \sqcap (s \downarrow\downarrow D) \tag{16}$$
$$(s \uparrow\downarrow C \sqcap D) \sqsubseteq (s \uparrow\downarrow C) \sqcap (s \uparrow\downarrow D) \tag{17}$$
$$(s \downarrow C \sqcap D) \equiv (s \downarrow C) \sqcap (s \downarrow D) \tag{18}$$
$$(s \downarrow\uparrow C \sqcap D) \sqsubseteq (s \downarrow\uparrow C) \sqcap (s \downarrow\uparrow D) \tag{19}$$
$$(s \uparrow\uparrow C \sqcap D) \sqsubseteq (s \uparrow\uparrow C) \sqcap (s \uparrow\uparrow D) \tag{20}$$
$$(s \uparrow C \sqcap D) \sqsubseteq (s \uparrow C) \sqcap (s \uparrow D) \tag{21}$$
$$(s \downarrow\downarrow C \sqcup D) \sqsupseteq (s \downarrow\downarrow C) \sqcup (s \downarrow\downarrow D) \tag{22}$$
$$(s \uparrow\downarrow C \sqcup D) \sqsupseteq (s \uparrow\downarrow C) \sqcup (s \uparrow\downarrow D) \tag{23}$$
$$(s \downarrow C \sqcup D) \sqsupseteq (s \downarrow C) \sqcup (s \downarrow D) \tag{24}$$
$$(s \downarrow\uparrow C \sqcup D) \sqsupseteq (s \downarrow\uparrow C) \sqcup (s \downarrow\uparrow D) \tag{25}$$
$$(s \uparrow\uparrow C \sqcup D) \equiv (s \uparrow\uparrow C) \sqcup (s \uparrow\uparrow D) \tag{26}$$
$$(s \uparrow C \sqcup D) \equiv (s \uparrow C) \sqcup (s \uparrow D) \tag{27}$$

Under an involutive negation, and an R-implication or the S-implication associated to the negation, we have that:

$$(s \downarrow C) \equiv \neg(s \uparrow \neg C) \tag{28}$$
$$(s \uparrow C) \equiv \neg(s \downarrow \neg C) \tag{29}$$
$$(s \downarrow\downarrow C) \equiv \neg(s \uparrow\uparrow \neg C) \tag{30}$$
$$(s \uparrow\uparrow C) \equiv \neg(s \downarrow\downarrow \neg C) \tag{31}$$
$$(s \uparrow\downarrow C) \equiv \neg(s \downarrow\uparrow \neg C) \tag{32}$$
$$(s \downarrow\uparrow C) \equiv \neg(s \uparrow\downarrow \neg C) \tag{33}$$

Under an R-implication, it holds that:

$$(s \uparrow\downarrow C) \sqsubseteq C \sqsubseteq (s \downarrow\uparrow C) \tag{34}$$

The following properties justify the fact that we used fuzzy similarity relations rather than fuzzy equivalence relations. Under an R-implication, if s is also transitive (i.e. if s is a fuzzy equivalence relation), it holds that:

$$(s \uparrow\downarrow (s \uparrow\downarrow C)) \equiv (s \uparrow\downarrow C) \tag{35}$$
$$(s \downarrow\uparrow (s \downarrow\uparrow C)) \equiv (s \downarrow\uparrow C) \tag{36}$$
$$(s \downarrow\downarrow C) \equiv (s \uparrow\downarrow C) \equiv (s \downarrow C) \tag{37}$$
$$(s \downarrow\uparrow C) \equiv (s \uparrow\uparrow C) \equiv (s \uparrow C) \tag{38}$$

Now we will introduce some new properties showing the relation between the fuzzy rough constructors and the universal and existential quantification:

$$(s \downarrow C) \equiv \forall s \cdot C \tag{39}$$
$$(s \uparrow C) \equiv \exists s \cdot C \tag{40}$$
$$(s \downarrow\downarrow C) \equiv (s \downarrow (s \downarrow C)) \tag{41}$$
$$(s \uparrow\downarrow C) \equiv (s \uparrow (s \downarrow C)) \tag{42}$$
$$(s \downarrow\uparrow C) \equiv (s \downarrow (s \uparrow C)) \tag{43}$$
$$(s \uparrow\uparrow C) \equiv (s \uparrow (s \uparrow C)) \tag{44}$$

Properties (39)–(44) show that fuzzy rough $\mathcal{SROIQ}$(D) can be reduced to (and hence it is not more expressive than) fuzzy $\mathcal{SROIQ}$(D). Furthermore, as we will see in Section 6, they are crucial to obtain our reasoning algorithms and to show that reasoning in both logics belong to the same complexity class.

There are several properties inherited from fuzzy $\mathcal{SROIQ}$(D) that are not discussed here [5,6,9]. We just recall that under an R-implication, there are several axioms which are syntactic sugar (and consequently it can be assumed that they do not appear in fuzzy KBs) due to the following equivalences [5]:

- $\mathtt{irr}(S) \equiv \top \sqsubseteq \neg\exists S.\mathtt{Self}$,
- $\mathtt{trans}(R) \equiv RR \sqsubseteq R$,
- $\mathtt{sym}(R) \equiv R \sqsubseteq R^-$.

4. Some use cases

In this section we present some use cases that demonstrate the usefulness of the fuzzy rough DLs proposed here. Section 4.1 discusses the application to a general notion of problems: information systems. Next, Section 4.2 presents the application to query refinement in information retrieval. Finally, Section 4.3 discusses the use of fuzzy rough concepts as concept modifiers. All these use cases show that our approach offer some advantages with respect to both classical ontologies and fuzzy rough set theory.

4.1. Information systems

An *information system* is a pair (U,A), where $U = \{o_1,\ldots,o_k\}$ is a non-empty finite set of objects (the *universe*) and $A = \{a_1,\ldots,a_m\}$ is a non-empty finite set of *attributes* such that $a : U \to V_a$ for every $a \in A$, where the set V_a is called the *value set* of a [34].

Consider a sample information system, originally proposed in [61], and defined as follows:

Object	a_1	a_2	a_3	a_4	a_5
o_1	1	2	0	1	1
o_2	1	2	0	1	1
o_3	2	0	0	1	0
o_4	0	0	1	2	1
o_5	2	1	0	2	1
o_6	0	0	1	2	2
o_7	2	0	0	1	0
o_8	0	1	2	2	1
o_9	2	1	0	2	2
o_{10}	2	0	0	1	0

Firstly, we would like to note that crisp information systems can trivially be represented by using a DL. Let v_{ij} be the value of the attribute a_j for the object o_i. For every cell of the information system, we can add a role assertion of the form:

$$(\mathsf{o}_i, \mathsf{v}_{ij}) : \mathsf{a}_j$$

According to the available information, the objects o_1 and o_2 cannot be distinguished. Similarly, the objects o_3, o_7, and o_{10} cannot be distinguished either. Note that a indiscernibility relation R can be explicitly represented in a KB as follows:

$$(\mathsf{o}_1, \mathsf{o}_2) : \mathsf{R}, (\mathsf{o}_3, \mathsf{o}_7) : \mathsf{R}, (\mathsf{o}_3, \mathsf{o}_{10}) : \mathsf{R}, (\mathsf{o}_7, \mathsf{o}_{10}) : \mathsf{R}$$

Furthermore, crisp information systems may be generalized to the case where there is some information about the application domain represented using a DL KB and, thus, attribute values can be inferred from the available knowledge. For instance, assume the following KB:

$$\mathcal{K} = \{\mathsf{fiat500} : \mathsf{Fiat}, \mathsf{Fiat} \sqsubseteq \mathsf{Car}, \mathsf{Car} \sqsubseteq \exists \mathsf{numberOfWheels}.\{2\}\}$$

It is easy to see that $\mathcal{K}$ entails the following role assertion, which means that the object fiat500 has a value 2 for the attribute numberOfWheels.

$$(\mathsf{fiat500}, 2) : \mathsf{numberOfWheels}$$

This approach can be extended to the fuzzy case. A *fuzzy information system* is a pair (U,A), where U is the universe and $A = \{a_1,\ldots,a_m\}$ is a non-empty finite set of *fuzzy attributes* such that $a : U \to \mathfrak{V}_a$ for every $a \in A$. Essentially, the values of a fuzzy attribute a (the values of the cells in a information system) are fuzzy sets. For instance, let us consider again the previous example, but now assuming that a_5 is a fuzzy attribute:

Object	a_1	a_2	a_3	a_4	a_5
o_1	1	2	0	1	High
o_2	1	2	0	1	Very high
o_3	2	0	0	1	Low
o_4	0	0	1	2	Medium
o_5	2	1	0	2	Low
o_6	0	0	1	2	Medium
o_7	2	0	0	1	Very low
o_8	0	1	2	2	High
o_9	2	1	0	2	Medium
o_{10}	2	0	0	1	Low

Let v_{ij} be a fuzzy set represented by one of the fuzzy membership functions allowed as fuzzy concrete predicates **d**. Then, every cell can be represented as follows:

$$\mathsf{o}_i : \exists \mathsf{a}_j \cdot \mathsf{v}_{ij}$$

For instance, the fact that fiat500 has a value medium for the attribute hasMaxSpeed (representing the maximum speed), can be represented as follows:

$$\mathsf{fiat500} : \exists \mathsf{hasMaxSpeed} \cdot \mathsf{Medium}$$

In the fuzzy case, the indiscernibility relation is fuzzy. For example, o_1 and o_2 are not fully indiscernible. On the contrary, they are indiscernible to some degree α_1, which depends on the compatibility of the labels *high* and *very high*. Again, this can be represented with the following fuzzy role assertion:

$$\langle(\mathsf{o}_1, \mathsf{o}_2) : \mathsf{R} \geqslant \alpha_1\rangle$$

Of course, attribute values can be inferred from the available knowledge. For instance, given the following KB:

$$\mathcal{K} = \{\langle \mathsf{porsche911} : \mathsf{FastCar} \geqslant \beta\rangle, \mathsf{FastCar} \sqsubseteq \exists \mathsf{hasMaxSpeed.High}\}$$

It is easy to see that $\mathcal{K}$ entails the following fuzzy concept assertion under an R-implication:

$$\langle \mathsf{porsche911} : \exists \mathsf{hasMaxSpeed} \cdot \mathsf{High} \geqslant \beta\rangle$$

In summary, the advantage of our approach compared to fuzzy rough theory is that we can have an ontology with background knowledge and use it to deduce implicit knowledge.

4.2. Query refinement in information retrieval

Query refinement is an essential task in information retrieval that aims to obtain different search results by extending a particular query with new terms that are related to the original query terms.

De Cock et al. studied the application of fuzzy rough sets to the query refinement problem [11,13]. In particular, they showed that the use of tight upper approximations works very well in practice, since it combines the effects of upper (that introduce too many irrelevant terms) and lower approximations (that easily produce an empty query).

In this subsection, we will show how to encode query refinement problems in fuzzy rough $\mathcal{SROIQ}(\mathsf{D})$ by presenting an example originally proposed in [11,13].

Let *R* be a fuzzy equivalence relation among all the possible query terms. *R* measures to which extent two terms can be considered as similar. For instance, *R* can be computed by measuring the co-occurrence frequency of pair of terms. The definition of *R* is expressed in the following table of fuzzy role assertions:

R	mac	computer	apple	fruit	pie	recipe	store	emulator	hardware
mac	1	0.89	0.89	0	0.01	0	0.75	0.83	0.66
computer		1	0.94	0.44	0.44	0.56	0.25	1	0.83
apple			1	0.83	0.99	0.83	0.83	0.25	0.99
fruit				1	0.44	0.66	1	0	0.03
pie					1	1	0.97	0	0.06
recipe						1	1	0	0.03
store							1	0.34	0.75
emulator								1	1
hardware									1

The degrees of truth are interpreted as lower bounds, except if the degree is 0, which is interpreted as an exact bound. For instance, the second cell in the table represents the fuzzy role assertion $\langle(\mathsf{mac},\mathsf{computer}):\mathsf{R}) \geqslant 0.89\rangle$, while the fourth cell represents the $\langle(\mathsf{mac},\mathsf{fruit}):\mathsf{R}=0\rangle$.

Now, let us consider a sample query. For instance, suppose that we are looking for apple pie recipes. Hence, we are interested in documents that have to do with all the terms apple, pie and recipe. Note that apple is an ambiguous word, which can refer both to a piece of fruit and to a computer company. This query can be seen as a conjunctive query (apple, pie, recipe). In our fuzzy rough DL, this query can be represented with the following assertions:

$$\langle \mathsf{apple} : \mathsf{Q} = 1\rangle, \quad \langle \mathsf{pie} : \mathsf{Q} = 1\rangle, \quad \langle \mathsf{recipe} : \mathsf{Q} = 1\rangle$$

It is important to stress that the original example is implicitly making a *closed domain assumption*. This is not usually done in DLs, but it can enforced by adding the following axiom:

$$\top \sqsubseteq \{\mathsf{mac}\} \sqcup \{\mathsf{computer}\} \sqcup \{\mathsf{apple}\} \sqcup \{\mathsf{fruit}\} \sqcup \{\mathsf{pie}\} \sqcup \{\mathsf{recipe}\} \sqcup \{\mathsf{store}\} \sqcup \{\mathsf{emulator}\} \sqcup \{\mathsf{hardware}\}$$

Once we have defined our fuzzy KB $\mathcal{K}$, the relevance of a term t to a query Q, can be computed using the new fuzzy rough constructors as follows:

$$glb(\mathcal{K}, \mathsf{t} : (\mathsf{R} \downarrow\uparrow \mathsf{Q}))$$

For instance, even if $glb(\mathcal{K}, \mathsf{mac} : \mathsf{Q}) = 0\rangle$, it holds that $glb(\mathcal{K}, \mathsf{mac} : (\mathsf{R} \downarrow\uparrow \mathsf{Q})) = 0.42$. That is, the query refinement process makes the term mac relevant to the query to some degree.

The following table allows to compare the results that we obtain with the original query (first column) and with the query refinement process based on the tight upper approximation (second column):

	Q	(R ↓↑ Q)
mac	0	0.42
computer	0	0.25
apple	1	1
fruit	0	0.83
pie	1	1
recipe	1	1
store	0	0.83
emulator	0	0.25
hardware	0	0.25

The interesting advantage of our approach is that we can represent not only the query terms and their relations, but also some background knowledge related to the domain, by using the different possibilities that fuzzy rough $\mathcal{SROIQ}(\mathsf{D})$ offers.

Additionally, we can trivially add weights to the query terms, and consider queries of the form $\{w_1/\mathsf{apple}, w_2/\mathsf{pie}, w_3/\mathsf{recipe}\}$, which can be represented as:

$$\langle \mathsf{apple} : \mathsf{Q} \geqslant w_1\rangle, \quad \langle \mathsf{pie} : \mathsf{Q} \geqslant w_2\rangle, \quad \langle \mathsf{recipe} : \mathsf{Q} \geqslant w_3\rangle$$

4.3. Fuzzy rough concepts as concept modifiers

To conclude this section, we would like to discuss the use of fuzzy rough concepts as concept modifiers. De Cock et al. proposed the use of fuzzy rough sets as fuzzy modifiers [14]. For example, the authors considered the modifiers roughly, moreOrLess, rather, very, definitely, and extremely, interpreted by means of the following inclusion relation:

$$\begin{aligned} \mathsf{extremely}(X) &\subseteq \mathsf{definitely}(X) \subseteq \mathsf{very}(X) \subseteq X \subseteq \mathsf{rather}(X) \\ &\subseteq \mathsf{moreOrLess}(X) \subseteq \mathsf{roughly}(X) \end{aligned} \tag{45}$$

Given a fuzzy concept C and a fuzzy similarity relation s, it is possible to define these fuzzy modifiers as follows:

$$\begin{aligned} &\mathsf{extremely}(\mathsf{C}) = (s \downarrow\downarrow \mathsf{C})_{\mathrm{KD}} \\ &\mathsf{definitely}(\mathsf{C}) = (s \downarrow \mathsf{C})_{\mathrm{KD}} \\ &\mathsf{very}(\mathsf{C}) = (s \downarrow \mathsf{C})_{\text{Ł}} \\ &\mathsf{rather}(\mathsf{C}) = (s \uparrow \mathsf{C})_{\text{Ł}} \\ &\mathsf{moreOrLess}(\mathsf{C}) = (s \uparrow \mathsf{C})_{\mathrm{G}} \\ &\mathsf{roughly}(\mathsf{C}) = (s \uparrow\uparrow \mathsf{C})_{\mathrm{G}} \end{aligned}$$

It is not difficult to see that using the previous definition, the inclusion relation (45) among modifiers holds.

Due to Property (7), it is immediate to see that in fact extremely(C) ⊆ definitely(C), very(C) ⊆ C ⊆ rather(C), moreOrLess(C) ⊆ roughly(C).

Furthermore, the well-known property $\alpha \otimes_Ł \beta \leqslant \alpha \otimes_G \beta$, implies that $(s \uparrow C)_Ł \subseteq (s \uparrow C)_G$. Consequently, rather(C) ⊆ moreOrLess(C).

Similarly, the well-known property $(\ominus\alpha) \oplus_G \beta \leqslant (\ominus\alpha) \oplus_Ł \beta$, implies that $\alpha \Rightarrow_{KD} \beta \leqslant \alpha \Rightarrow_Ł \beta$. Consequently $(s \downarrow C)_{KD} \subseteq (s \downarrow C)_Ł$ and thus definitely(C) ⊆ very(C).

An advantage of this approach for modelling fuzzy concept modifiers is the flexibility, since a fuzzy ontology designer can count on different fuzzy rough approximations and different fuzzy logic operators. It is also interesting to remark that fuzzy rough constructors are implemented in some fuzzy DL reasoners, as described in Section 5. In the literature there are several approaches for representing fuzzy modifiers in fuzzy DLs [16,25,26,55,60,62]. However, they have not been implemented in practice with the notable exception of triangular and linear fuzzy modifiers, supported by fuzzyDL [7] and DeLorean [4].

The advantage of our approach compared to fuzzy rough theory is that we can apply these concept modifiers to complex concepts built using the constructors of the fuzzy rough DL.

5. Reasoning and implementation

In this section we will show how two current highly expressive fuzzy DL reasoners have been adapted to support fuzzy rough DLs, namely fuzzyDL system [7] and DeLorean system [4].

To this end, we recall that Properties (39) and (40) show that we can map upper $(s_i \uparrow C)$ and lower $(s_i \downarrow C)$ approximation concepts can be represented as fuzzy DL concepts $\exists s_i \cdot C$ and $\forall s_i \cdot C$, respectively. That is, we consider the following transformation:

$$(s_i \downarrow C) \mapsto \forall s_i \cdot C \tag{46}$$

$$(s_i \uparrow C) \mapsto \exists s_i \cdot C \tag{47}$$

Thus, we may replace upper and lover approximation concepts with ordinary fuzzy DL concepts. This is exactly the same transformation pointed out for the crisp case [50].

Furthermore, tight and loose approximation are translated by using Properties (41)–(44):

$$(s \downarrow\downarrow C_i) \mapsto \forall s_i \cdot (\forall s_i \cdot C_i) \tag{48}$$

$$(s \uparrow\downarrow C_i) \mapsto \exists s_i \cdot (\forall s_i \cdot C_i) \tag{49}$$

$$(s \downarrow\uparrow C_i) \mapsto \forall s_i \cdot (\exists s_i \cdot C_i) \tag{50}$$

$$(s \uparrow\uparrow C_i) \mapsto \exists s_i \cdot (\exists s_i \cdot C_i) \tag{51}$$

We also need to add some axioms stating that s_i is a fuzzy similarity or fuzzy equivalence relation. Since the translation can be performed in polynomial time and introduces a polynomial number of new axioms (2 for every fuzzy similarity relation and 3 for every fuzzy equivalence relation), reasoning in fuzzy rough $\mathcal{SROIQ}(\mathbf{D})$ and reasoning in fuzzy $\mathcal{SROIQ}(\mathbf{D})$ belong to the same complexity class. In general, the complexity class for reasoning with fuzzy $\mathcal{SROIQ}(\mathbf{D})$ is still unknown. For some fragments of it, reductions to the non-fuzzy case are known [5,6,9].

In the following, we shall discuss some concrete implementation details.

5.1. *fuzzyDL reasoner*

fuzzyDL is a publicly available reasoner for fuzzy $\mathcal{SHIF}(\mathbf{D})$ under Zadeh, Łukasiewicz and Gödel logics [7]. The differences with respect to the logic defined in Section 3 are the following. On the one hand, it did not initially support role constructors R4–R6, concept constructors C12–C16, C22–C27, and axioms A3, A5, A12–A15, A17. On the other hand, fuzzyDL can also support a lot of features not covered here.

Its reasoning algorithm combines a tableaux algorithm and a mixed integer linear optimization problem. The basic idea is to build a tableaux using a set of satisfiability preserving rules which generate new simpler fuzzy assertion axioms together with some inequations over [0, 1]-valued variables. Finally, an optimization problem through the set of inequations is solved. A detailed description of the reasoning algorithm cannot fit into this paper, but it can be found in [58].

In this subsection we describe how we have extended fuzzyDL in order to support C16, C22–C27, and A14. Essentially, we needed to support reflexive roles, added some convenient syntactic sugar, and implemented the transformation in Eqs. (46)–(51).

Firstly, we included some new concept constructors: tight lower approximations, loose lower approximation, (conventional) lower approximations, tight upper approximations, loose upper approximation, (conventional) upper approximations, and local reflexivity concepts, which are of the following forms: `(tla` s_i C), `(lla` s_i C), `(la` s_i C), `(tua` s_i C), `(lua` s_i C), `(ua` s_i C), `(self` S), respectively, where s_i is a fuzzy similarity or a fuzzy equivalence relation, S is a simple fuzzy role and C is a fuzzy concept. Local reflexivity concepts are not necessary for the rough extension, but adding them is easy (reasoning is similar to the case of reflexive roles).

Fuzzy similarity and equivalence relations must be previously defined using the following syntax:

(define − fuzzy − similarity s_i).

(define − fuzzy − equivalence s_i).

Then, we extended FUZZYDL with reflexive roles of the form (reflexive R), where R is a fuzzy role. For convenience, we allowed symmetric role axioms of the form (symmetric R). Symmetric role axioms are just syntactic sugar and could already be simulated in FUZZYDL, since, under an R-implication, the axiom $R \sqsubseteq R^-$ and the symmetry of R are equivalent.

The reasoning algorithm was extended as follows:

- For every fuzzy similarity relation (define-fuzzy-similarity s_i) we assert s_i to be reflexive and symmetric by adding the following axioms: (reflexive R), (symmetric R).
- For every fuzzy equivalence relation (define-fuzzy-equivalence s_i), we assert s_i to be reflexive, and symmetric (as in the previous step), but also transitive by adding the following axiom: (transitive R).
- Every tight lower approximation concept (tla s_i C) is replaced with a lower restriction concept (la (la s_i C)).
- Every loose lower approximation concept (lla s_i C) is replaced with an upper restriction concept (ua(la s_i C)).
- Every tight upper approximation concept (tua s_i C) is replaced with a lower restriction concept (la(ua s_i C)).
- Every loose upper approximation concept (lua s_i C) is replaced with an upper restriction concept (ua(ua s_i C)).
- Every lower approximation concept (la s_i C) is replaced with a universal restriction concept (all s_i C).
- Every upper approximation concept (ua s_i C) is replaced with an existential restriction concept (some s_i C).
- Every symmetric role axiom (symmetric R) is replaced with an inverse role axiom (inverse R invR) and a role inclusion axiom (implies-role R invR). Under an R-implication, it is well known that sym(R) is equivalent to $R \sqsubseteq R^-$.
- The rule for a local reflexivity concept (self S) asserts that an individual is related to itself. Roughly speaking, for every individual x of a model, we create a relation $S(x,x)$ with some degree.
- The rule for reflexive roles (reflexive R) asserts that every individual is related to itself. Roughly speaking, for every individual x of a model, we create a relation $R(x,x) = 1$.

5.2. DeLorean reasoner

DeLorean is a reasoner for fuzzy $\mathcal{SROIQ}$(D) under Zadeh and Gödel logics [4]. The differences with respect to the logic defined in Section 3 are that it did not initially support role constructors R5–R6, and concept constructors C17–C18 and C22–C27.

Its reasoning algorithm is based on a reduction to a classical DL, so current DL reasoners can be reused. A full description may be found in [5,6].

In this subsection we describe how we have extended DeLorean in order to support C22–C27. Essentially, we implemented the transformation in Eqs. (46)–(51).

To begin with, we extended the syntax with the new concept constructors[5]: (tight-lower s_i C), (loose-lower s_i C), (lower s_i C), (tight-upper s_i C), (loose-upper s_i C), (upper s_i C), where s_i is a fuzzy similarity relation and C is a fuzzy concept.

Then, the reasoning algorithm was extended as follows:

- Every tight lower approximation concept (tight-lower s_i C) is replaced with a lower restriction concept (lower (lower s_i C)).
- Every loose lower approximation concept (loose-lower s_i C) is replaced with an upper restriction concept (upper(-lower s_i C)).
- Every tight upper approximation concept (tight-upper s_i C) is replaced with a lower restriction concept (lower (upper s_i C)).
- Every loose upper approximation concept (lower—upper s_i C) is replaced with an upper restriction concept (upper(upper s_i C)).
- Every concept (upper s_i C) is replaced with an existential restriction concept (some s_i C). Furthermore, we add the following axioms if they do not exist in the fuzzy RBox: (reflexive s_i), (symmetric s_i).
- Every concept (lower s_i C) is replaced with a universal restriction concept (all s_i C). Once again, we add the following axioms in case they do not exist in the fuzzy RBox: (reflexive s_i), (symmetric s_i).

Note that the fuzzy relations that take part in some approximation concept are interpreted as fuzzy similarity relations. If the user wants to interpret a fuzzy relation s as a fuzzy equivalence relation, it is necessary to add a transitive role axiom (transitive s).

6. Related work

In this section, we will begin by discussing separately fuzzy and rough extensions of DLs, and then we will report on the few works that consider fuzzy rough DLs.

[5] The syntax is similar to that of FUZZYDL, although there are some differences in the reserved words of the language for historical reasons.

Fuzzy DLs. Since the first work of Yen in 1991 [64], an important number of fuzzy extensions to DLs can be found in the literature. Some notable works are [5,6,8,9,21,24,47,52,53,55–57,60]; for a more detailed survey we refer the reader to the survey in [38]. The present paper builds on the very expressive fuzzy DL $\mathcal{SROIQ}(\mathsf{D})$, extending it with fuzzy rough sets.

Rough DLs. There are also some works on rough DLs [17,19,30,33,32,36,50]. In particular, Schlobach et al. show that for any DL having universal and existential restrictions, as well as reflexive, symmetric and transitive roles, reasoning in a rough DL can be reduced to reasoning with a classical DL [50]. In the present paper, however, we introduce rough concepts in a fuzzy DL, instead of in a classical DL.

Fuzzy rough DLs. There is little research on fuzzy rough DLs. In the older paper, Dey et al. present a fuzzy rough ontology [15]. They do not elaborate on the formal details of the subjacent DL, and reasoning is restricted to a simple form of concept querying. Our approach however, has a more formal basis and makes it possible to use any of the reasoning tasks for fuzzy DLs.

Jiang et al. provide a fuzzy rough extension of the DL $\mathcal{SHIN}$, and show how that it is equivalent to fuzzy $\mathcal{SHIN}$ by using a similar translation as in the current paper [29]. Jiang et al. also provide an extension of the DL $\mathcal{ALC}$ with intuitionistic fuzzy rough sets [28]. Although a useful syntactic sugar is provided, their approach does not provide additional expressive power, since they also show how to reduce reasoning in their logic to reasoning in fuzzy $\mathcal{ALC}$ over a lattice [56]. These works are restricted to Zadeh logic, and consider generalized fuzzy rough sets. In generalized fuzzy rough sets, indiscernibility relations are not required to be reflexive and symmetric. This assumption of generalized fuzzy rough sets is necessary because reflexivity cannot be represented neither in $\mathcal{ALC}$ nor in $\mathcal{SHIN}$.

There are several differences comparing to our approach. Firstly, we consider the more expressive DL $\mathcal{SROIQ}(\mathsf{D})$ as the subjacent logic. Secondly, as a consequence of the increase of expressivity, indiscernibility relations can be represented using fuzzy similarity or fuzzy equivalence relations. Thirdly, the semantics of our logic is independent from the fuzzy logic considered. In particular, we allow to use an R-implication, which is necessary to prove some of the common properties of fuzzy rough sets (for some background on implication functions, the reader is referred to Section 2.1). On the contrary [29] uses Kleene–Dienes implication, which is known to have counter-intuitive effects in fuzzy DLs [5,24]. Fourthly, we have introduced tight and loose rough approximations [13]. Finally, we have described how to implement our approach.

7. Conclusions

In this paper we have studied a DL managing vagueness in two different but complementary ways, combining a fuzzy DL with fuzzy rough sets. In particular, we have presented a very expressive fuzzy rough extension of the DL $\mathcal{SROIQ}(\mathsf{D})$, the logic behind the language OWL 2. The rough extension is general (independent of the family of fuzzy operators), and uses k possible fuzzy similarity relations. In addition to the well-known lower and upper approximations, we include tight and loose approximations, reflecting the fact that an element can belong to several fuzzy similarity classes.

Reasoning in fuzzy rough $\mathcal{SROIQ}(\mathsf{D})$ can be reduced to reasoning in fuzzy $\mathcal{SROIQ}(\mathsf{D})$. Reasoning under this latter logic is not currently possible, but we have extended and implemented two well-known reasoning algorithms for fuzzy DLs in order to deal with two important fragments of the logic. On the one hand, FUZZYDL implements a combination of a tableaux algorithm and a mixed integer linear optimization problem, and already supports fuzzy rough $\mathcal{SHIF}(\mathsf{D})$ under Zadeh, Łukasiewicz and Gödel logics. On the other hand, DELOREAN implements a translation to a crisp DL and supports fuzzy rough $\mathcal{SROIQ}(\mathsf{D})$ under Zadeh and Gödel logics.

Extending the expressivity of the fuzzy DLs reasoners, in order to support in practice more expressive fragments of fuzzy rough $\mathcal{SROIQ}(\mathsf{D})$, remains an open research problem.

Another approach that would be worth to explore is the extension of rough and fuzzy rough DLs with possibilistic logic, similarly as in [42], where degrees of certainty indicate to what extent equivalence classes are included in (possibly fuzzy) rough concepts.

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References

[1] Franz Baader, Diego Calvanese, Deborah McGuinness, Daniele Nardi, Peter F. Patel-Schneider, The Description Logic Handbook: Theory, Implementation, and Applications, Cambridge University Press, 2003.
[2] Franz Baader, Ian Horrocks, Ulrike Sattler, Description logics as ontology languages for the semantic web, in: Mechanizing Mathematical Reasoning: Essays in Honor of Jörg H. Siekmann on the Occasion of His 60th Birthday, Lecture Notes in Artificial Intelligence, vol. 2605, Springer-Verlag, 2005, pp. 228–248.
[3] Tim Berners-Lee, James Hendler, Ora Lassila, The semantic web, Scientific American 284 (5) (2001) 34–43.

[4] Fernando Bobillo, Miguel Delgado, Juan Gómez-Romero, DeLorean: A reasoner for fuzzy OWL 2.Expert Systems with Applications 39 (1) (2012) 258–272.

[5] Fernando Bobillo, Miguel Delgado, Juan Gómez-Romero, Crisp representations and reasoning for fuzzy ontologies, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 17 (4) (2009) 501–530.

[6] Fernando Bobillo, Miguel Delgado, Juan Gómez-Romero, Umberto Straccia, Fuzzy description logics under Gödel semantics, International Journal of Approximate Reasoning 50 (3) (2009) 494–514.

[7] Fernando Bobillo, Umberto Straccia, fuzzyDL: an expressive fuzzy description logic reasoner, in: Proceedings of the 17th IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2008), IEEE Computer Society, 2008, pp. 923–930.

[8] Fernando Bobillo, Umberto Straccia, Fuzzy description logics with general t-norms and datatypes, Fuzzy Sets and Systems 160 (23) (2009) 3382–3402.

[9] Fernando Bobillo, Umberto Straccia, Reasoning with the Łukasiewicz fuzzy description logic $\mathcal{SROIQ}$, Information Sciences 181 (2011) 758–778.

[10] Fernando Bobillo, Umberto Straccia, Supporting fuzzy rough sets in fuzzy description logics, in: Proceedings of the 9th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2009), Lecture Notes in Computer Science, vol. 5590, Springer-Verlag, 2009, pp. 676–687.

[11] Chris Cornelis, Martine De Cock, Anna Maria Radzikowska, Fuzzy rough sets: from theory into practice, in: Witold Pedrycz, Andrzej Skowron, Vladik Kreinovich (Eds.), Handbook of Granular Computing, Wiley-Interscience, 2008, pp. 533–552.

[12] Bernardo Cuenca-Grau, Ian Horrocks, Boris Motik, Bijan Parsia, Peter F. Patel-Schneider, Ulrike Sattler, OWL 2: the next step for OWL, Journal of Web Semantics 6 (4) (2008) 309–322.

[13] Martine De Cock, Chris Cornelis, Etienne E. Kerre, Fuzzy rough sets: the forgotten step, IEEE Transactions on Fuzzy Systems 15 (1) (2007) 121–130.

[14] Martine De Cock, Anna Maria Radzikowska, Etienne E. Kerre, A fuzzy-rough approach to the representation of linguistic hedges, in: Bernadette Bouchon-Meunier, Julio Gutiérrez-Ríos, Luis Magdalena, Ronald R. Yager (Eds.), Technologies for Constructing Intelligent Systems I, Studies in Fuzziness and Soft Computing, vol. 89, Springer-Verlag, 2002, pp. 33–42.

[15] Lipika Dey, Muhammad Abulaish, Rohit Goyal, Kumar Shubham, A rough-fuzzy ontology generation framework and its application to bio-medical text processing, in: Proceedings of the 5th Atlantic Web Intelligence Conference (AWIC 2007), Advances in Soft Computing, vol. 43, Springer-Verlag, 2007, pp. 74–79.

[16] Dzung Dinh-Khac, Dinh-Khang Tran, Steffen Hölldobler, The fuzzy linguistic description logic $\mathcal{ALC}_{\mathcal{FL}}$, in: Proceedings of the 11th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU 2006), 2006, pp. 2096–2103.

[17] Patrick Doherty, Michał Grabowski, Witold Łukaszewicz, Andrzej Szałas, Towards a framework for approximate ontologies, Fundamenta Informaticae 57 (2–4) (2003) 147–165.

[18] Didier Dubois, Henri Prade, Rough fuzzy sets and fuzzy rough sets, International Journal of General Systems 17 (2–3) (1990) 191–209.

[19] Nicola Fanizzi, Claudia d'Amato, Floriana Esposito, Thomas Lukasiewicz, Representing uncertain concepts in rough description logics via contextual indiscernibility relations, in: Proceedings of the 4th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2008), CEUR Workshop Proceedings, vol. 423, 2008.

[20] Thomas R. Gruber, A translation approach to portable ontologies, Knowledge Acquisition 5 (2) (1993) 199–220.

[21] Volker Haarslev, Hsueh-Ieng Pai, Nematollaah Shiri, A formal framework for description logics with uncertainty, International Journal of Approximate Reasoning 50 (9) (2009) 1399–1415.

[22] Reiner Hähnle, Advanced many-valued logics, in: Dov M. Gabbay, F. Guenthner (Eds.), Handbook of Philosophical Logic, 2nd ed., vol. 2, Kluwer, 2001.

[23] Petr Hájek, Metamathematics of Fuzzy Logic, Kluwer, 1998.

[24] Petr Hájek, Making fuzzy description logic more general, Fuzzy Sets and Systems 154 (1) (2005) 1–15.

[25] Steffen Hölldobler, Hoang-Nga Nguyen, Dinh-Khang Tran, The fuzzy description logic $\mathcal{ALC}_{\mathcal{FLH}}$, in: Proceedings of the International Workshop on Description Logics (DL 2005), CEUR Workshop Proceedings 147, 2005.

[26] Steffen Hölldobler, Hans-Peter Störr, Dinh-Khang Tran, The fuzzy description logic $\mathcal{ALC}_{\mathcal{FH}}$ with hedge algebras as concept modifiers, Journal of Advanced Computational Intelligence and Intelligent Informatics 7 (3) (2003) 294–305.

[27] Ian Horrocks, Oliver Kutz, Ulrike Sattler, The even more irresistible $\mathcal{SROIQ}$, in: Proceedings of the 10th International Conference of Knowledge Representation and Reasoning (KR 2006), 2006, pp. 452–457.

[28] Yuncheng Jiang, Yong Tang, Ju Wang, Suqin Tang, Reasoning within intuitionistic fuzzy rough description logics, Information Sciences 179 (14) (2009) 2362–2378.

[29] Yuncheng Jiang, Ju Wang, Peimin Deng, Suqin Tang, Reasoning within expressive fuzzy rough description logics, Fuzzy Sets and Systems 160 (23) (2009) 3403–3424.

[30] Yuncheng Jiang, Ju Wang, Suqin Tang, Bao Xiao, Reasoning with rough description logics: an approximate concepts approach, Information Sciences 179 (5) (2009) 600–612.

[31] Yevgeny Kazakov, $\mathcal{SRIQ}$ and $\mathcal{SROIQ}$ are harder than $\mathcal{SHOIQ}$, in: Proceedings of the 11th International Conference on Principles of Knowledge Representation and Reasoning (KR 2008), AAAI Press, 2008, pp. 274–284.

[32] C. Maria Keet, On the feasibility of description logic knowledge bases with rough concepts and vague instances, in: Proceedings of the 23rd International Workshop on Description Logics (DL2010), CEUR Workshop Proceedings, vol. 573, 2010, pp. 314–324.

[33] Michel C.A. Klein, Peter Mika, Stefan Schlobach, Rough description logics for modeling uncertainty in instance unification, in: Proceedings of the 3rd ISWC Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2007), vol. 327, CEUR Workshop Proceedings, 2007.

[34] Jan Komorowski, Zdzisław Pawlak, Lech Polkowski, Andrzej Skowron, Rough sets: a tutorial, in: Sankar K. Pal, Andrzej Skowron (Eds.), Rough Fuzzy Hybridization A New Trend in Decision Making, Springer, 1999, pp. 3–98.

[35] Ludmila Ilieva Kuncheva, Fuzzy rough sets: application to feature selection, Fuzzy Sets and Systems 51 (2) (1992) 147–153.

[36] Churn-Jung Liau, On rough terminological logics, in: Proceedings of the 4th International Workshop on Rough Sets, Fuzzy Sets and Machine Discovery (RSFD'96), vol. 11, 1996, pp. 47–54.

[37] T.Y. Lin, Topological and fuzzy rough sets, in: R. Slowinski (Ed.), Decision Support by Experience: Applications of the Rough Sets Theory, Kluwer, 1992, pp. 287–304.

[38] Thomas Lukasiewicz, Umberto Straccia, Managing uncertainty and vagueness in description logics for the semantic web, Journal of Web Semantics 6 (4) (2008) 291–308.

[39] Nehad N. Morsi, M.M. Yakout, Axiomatics for fuzzy rough sets, Fuzzy Sets and Systems 100 (1–3) (1998) 327–342.

[40] Paul S. Mostert, Allen L. Shields, On the structure of semigroups on a compact manifold with boundary, Annals of Mathematics 65 (1) (1957) 117–143.

[41] A. Nakamura, Fuzzy rough sets, Note on Multiple-Valued Logic in Japan 9 (8) (1988) 1–8.

[42] Michinori Nakata, Hiroshi Sakai, Applying rough sets to information tables containing possibilistic values, Transactions on Computational Science 2 (2008) 180–204.

[43] S. Nanda, S. Majumdar, Fuzzy rough sets, Fuzzy Sets and Systems 45 (2) (1992) 157–160.

[44] Vilém Novák, Which logic is the real fuzzy logic?, Fuzzy Sets and Systems 157 (5) (2006) 635–641.

[45] Zdzisław Pawlak, Rough sets, International Journal of Computer and Information Sciences 11 (5) (1982) 341–356.

[46] Anna Maria Radzikowska, Etienne E. Kerre, A comparative study of fuzzy rough sets, Fuzzy Sets and Systems 126 (2) (2002) 137–155.

[47] Daniel Sánchez, Andrea G.B. Tettamanzi, Fuzzy quantification in fuzzy description logics, in: Elie Sanchez (Ed.), Fuzzy Logic and the Semantic Web, Capturing Intelligence, vol. 1, Elsevier Science, 2006, pp. 135–159.

[48] Elie Sanchez (Ed.), Fuzzy Logic and the Semantic Web, Capturing Intelligence, vol. 1, Elsevier Science, 2006.

[49] A. Schaerf, Reasoning with individuals in concept languages, Data and Knowledge Engineering 13 (2) (1994) 141–176.

[50] S. Schlobach, M.C.A. Klein, L. Peelen, Description logics with approximate definitions: precise modeling of vague concepts, in: Proceedings of the 20th International Joint Conference on Artificial Intelligence (IJCAI 2007), 2007, pp. 557–562.
[51] M. Schmidt-Schauß, G. Smolka, Attributive concept descriptions with complements, Artificial Intelligence 48 (1) (1991) 1–26.
[52] Giorgos Stoilos, Giorgos Stamou, Jeff Z. Pan, Vassilis Tzouvaras, Ian Horrocks, Reasoning with very expressive fuzzy description logics, Journal of Artificial Intelligence Research 30 (8) (2007) 273–320.
[53] Giorgos Stoilos, Giorgos Stamou, Vassilis Tzouvaras, Jeff Z. Pan, Ian Horrocks, Fuzzy OWL: uncertainty and the semantic web, in: Proceedings of the 1st International Workshop on OWL: Experience and Directions (OWLED 2005), CEUR Workshop Proceedings, vol. 188, 2005.
[54] Umberto Straccia, Reasoning within fuzzy description logics, Journal of Artificial Intelligence Research 14 (2001) 137–166.
[55] Umberto Straccia, Description logics with fuzzy concrete domains, in: Proceedings of the 21st Conference on Uncertainty in Artificial Intelligence (UAI 2005), AUAI Press, 2005.
[56] Umberto Straccia, Description logics over lattices, International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems 14 (1) (2006) 1–16.
[57] Umberto Straccia, A fuzzy description logic for the semantic web, in: Elie Sanchez (Ed.), Fuzzy Logic and the Semantic Web, Capturing Intelligence, vol. 1, Elsevier Science, 2006, pp. 73–90.
[58] Umberto Straccia, Reasoning in Ł-$\mathcal{SHIF}$: an expressive fuzzy description logic under Łukasiewicz semantics, Technical Report ISTI-018/2007, ISTI-CNR, 2007.
[59] Helmut Thiele, Fuzzy rough sets versus rough fuzzy sets – An interpretation and a comparative study using concepts of modal logic, Technical report, University of Dortmund, 1998.
[60] Christopher B. Tresp, Ralf Molitor, A description logic for vague knowledge, in: Proceedings of the 7th European Conference of Artificial Intelligence (ECAI 1998), 1998.
[61] Wikipedia. Rough set. <http://www.en.wikipedia.org/wiki/Rough_set>.
[62] Yining Wu, Transforming fuzzy description logic $\mathcal{ALC}_{\mathcal{FL}}$ into classical description logic $\mathcal{ALCH}$, in: Proceedings of the 6th International Workshop on Uncertainty Reasoning for the Semantic Web (URSW 2010), CEUR Workshop Proceedings 654, 2010, pp. 13–24.
[63] Y.Y. Yao, Combination of rough and fuzzy sets based on α-level sets, in: T.Y. Lim, N. Cercone (Eds.), Rough Sets and Data Mining: Analysis for Imprecise Data, Kluwer, 1997, pp. 301–321.
[64] John Yen, Generalizing term subsumption languages to fuzzy logic, in: Proceedings of the 12th International Joint Conference on Artificial Intelligence (IJCAI 1991), Morgan Kaufmann, 1991, pp. 472–477.
[65] Lotfi A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.